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TOPICAL REVIEW

Random walks in random environments

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Abstract
Random walks in random environments and their diffusion analogues have been a source of surprising phenomena and challenging problems, especially in the non-reversible situation, since they began to be studied in the 1970s. We review the model, available results and techniques, and point out several gaps in the understanding of these processes.

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1. Introduction

Random walks and their scaling limits, diffusion processes, provide a simple yet powerful description of random processes, and are fundamental in the description of many fields, from biology through economics, engineering, and statistical mechanics. A large body of work has accumulated concerning the properties of such processes, and very detailed information is available. We refer to [Sp76, L91, ReY99] and [StV79] for the background on random walks and diffusion processes.

Yet, in many situations, the medium in which the process evolves is highly irregular. Without further modelling, this results with spatially inhomogeneous Markov processes, and not much can be said. Things are however different if some degree of homogeneity is assumed on the *law* of the environment. When the underlying state space on which the walk moves with nearest-neighbour steps is the lattice $\mathbb{Z}^d$, $d \geqslant 1$, and the law of the environment is assumed stationary, we call the resulting random walk a *random walks in random environment* (RWRE). An effort to model such situations for random walks on $\mathbb{Z}$, originally motivated by biological applications, can be traced back to [T72]. We refer to [Hg96] for a comprehensive description of the literature up to 1996; see also [Rv05].

A precise formulation of the RWRE model is as follows. Let S denote the 2d-dimensional simplex, set $\Omega = S^{Z^d}$, and let $\omega(z,\cdot) = \{\omega(z, z+e)\}_{e\in Z^d,|e|=1}$ denote the coordinate of $\omega \in \Omega$

corresponding to $z \in Z^d$. ω is an 'environment' for an inhomogeneous nearest-neighbour random walk (RWRE) started at x with *quenched* transition probabilities $P_\omega(X_{n+1} = z + e|X_n = z) = \omega(x, x+e)$ $(e \in \mathbb{Z}^d, |e| = 1)$, whose law is denoted by P_ω^x. We write E_ω^x (and not $\langle\cdot\rangle_{P_\omega^x}$) for expectations with respect to the law P_ω^x, and write P_ω and E_ω for P_ω^0 and E_ω^0. In the RWRE model, the environment is random, of law P, which is always assumed stationary and ergodic. We often assume that the environment is *uniformly elliptic*, that is there exists an $\epsilon > 0$ such that P-a.s., $\omega(x, x+e) \geqslant \epsilon$ for all $x, e \in \mathbb{Z}^d$, $|e| = 1$. Finally, we denote by $\mathbb{P}$ the *annealed* law of the RWRE started at 0, that is the law of $\{X_n\}$ under the measure $P \times P_\omega^0$, and again we write $\mathbb{E}$ (and not $\langle\cdot\rangle_{\mathbb{P}}$) for expectations with respect to $\mathbb{P}$ and E (and not $\langle\cdot\rangle_P$) for expectations with respect to P. For future reference, we recall that, given a probability measure Q, a statement occurs Q almost surely (in short, Q-a.s.) if the Q-probability that the statement does not hold is 0. If a statement involving only the random walk holds $\mathbb{P}$-a.s., it implies that for P-almost every ω, the statement holds P_ω-a.s.

Mathematically, and especially for $d > 1$, the RWRE model leads to the analysis of irreversible, inhomogeneous Markov chains, to which standard tools of homogenization theory do not apply well. Further, unusual phenomena, such as sub-diffusive behaviour, polynomial decay of probabilities of large deviations, and trapping effects, arise, already in the one-dimensional model.

To get an idea of some of the unusual features of the RWRE model, we begin by discussing the one-dimensional case. This model, being reversible, is fairly well understood, and we review the results (in section 2) and available techniques (in section 3). We then turn in section 4 to the multidimensional non-reversible case in the non-perturbative regime. Section 5 is devoted to the description of some of the tools that have been developed in recent years to handle this situation, while section 6 is devoted to the perturbative regime. Section 7 quickly reviews the available results for the related model of (non-reversible) diffusions in random environments. In section 8, we collect some information about related models that we do not describe in detail in this review. This review borrows heavily from [Sz04, Zt02] and [Zt04].

2. One-dimensional RWRE

When $d = 1$, we write $\omega_x = \omega(x, x+1)$, $\rho_x = (1-\omega_x)/\omega_x$ and $u = E \log \rho_0$. The motion of the RWRE in the random environment resembles the motion of a particle in a random potential, where the potential at the point $x > 0$ is $V(x) = \sum_{i=0}^{x} \log \rho_i$. Thus, fluctuations in the environment that result in high potential barriers may confine the particle. We describe in this section the behaviour of the RWRE, postponing to section 3 a description of the analogy between the motion of particles in a random potential and the RWRE.

We recall some standard notation and definitions: for any sequence (a_n), $\sup a_n$ denotes the supremum of the sequence, i.e. the smallest number A such that $a_n \leqslant A$ for all n. If the sequences possess a maximum, its supremum equals that maximum. The infimum of the sequence, denoted by $\inf a_n$, equals $\sup(-a_n)$. Further, $\limsup a_n$, the limsup of (a_n), is the largest number A such that for any ϵ one can find a sequence $n_k \to \infty$ with $a_{n_k} > A - \epsilon$ for all k. Finally, $\liminf a_n = \limsup(-a_n)$.

2.1. Ergodic behaviour

Recall that for a homogeneous environment (that is, when the stationary measure P has a marginal which charges a single value: $\omega_i = \bar{\omega}$ for all i), we have $X_n/n \to v_{\bar{\omega}} := 2\bar{\omega} - 1$ and $(X_n - nv_{\bar{\omega}})/\sqrt{\bar{\omega}(1-\bar{\omega})n}$ converges in distribution to a standard Gaussian. Our first goal is to clarify the corresponding statements in the case of the RWRE, and in particular reveal some of the surprising phenomena associated with the RWRE. As it turns out, the sign of u determines the direction of escape of the RWRE, while the limiting behaviour depends on an explicit function of the law of the environment.

Theorem 2.1 (transience, recurrence, limit speed, $d = 1$)**.**

(a) *If $u < 0$ then $X_n \to_{n\to\infty} \infty$, $\mathbb{P}$-a.s. If $u > 0$ then $X_n \to -\infty$, $\mathbb{P}$-a.s. Finally, if $u = 0$ then the RWRE oscillates, that is, $\mathbb{P}$-a.s.,*

$$\limsup_{n\to\infty} X_n = \infty, \qquad \liminf_{n\to\infty} X_n = -\infty.$$

Further, there is a deterministic v such that

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \qquad \mathbb{P} - \text{a.s.}, \tag{2.1}$$

$v > 0$ if $\sum_{i=1}^{\infty} E\big(\prod_{j=0}^{i} \rho_{-j}\big) < \infty$, $v < 0$ if $\sum_{i=1}^{\infty} E\big(\prod_{j=0}^{i} \rho_{-j}^{-1}\big) < \infty$, and $v = 0$ if both these conditions do not hold.

(b) *If P is a product measure then*

$$v = \begin{cases} (1 - E(\rho_0))/(1 + E(\rho_0)), & E(\rho_0) < 1, \\ -\big(1 - E\big(\rho_0^{-1}\big)\big)/\big(1 + E\big(\rho_0^{-1}\big)\big), & E\big(\rho_0^{-1}\big) < 1, \\ 0, & \text{else}. \end{cases} \tag{2.2}$$

Statement (2.1) that X_n/n converges to a deterministic limit (under both the quenched and annealed measures) is referred to as a law of large numbers (LLN). Theorem 2.1 is essentially due to [So75]; see [A99] and [Zt04] for a proof in the general ergodic setup. In sections 3.1 and 3.2, we sketch the argument.

Remark 2.2. The surprising features of the RWRE model can be appreciated if one notes the following facts, all for a product measure P:

(a) Suppose $u < 0$, that is $X_n \to \infty$, $\mathbb{P}$-a.s. By Jensen's inequality, $\log E\rho_0 \geqslant E \log \rho_0$, but it is quite possible that $E\rho_0 > 1$. Thus, it is possible to construct i.i.d. environments in which the RWRE is transient, but the speed $v = 0$.

(b) Suppose $\bar{v} = 2E\omega_0 - 1$ denotes the speed of a (biased) simple random walk with probability of jump to the right equal, at any site, to $E\omega_0$. It is easy to construct examples with $\bar{v} > 0$ but $u > 0$, which means that $X_n \to -\infty$ even if the static speed $\bar{v}$ points to the right (such an example is obtained if, for example, ω_0 equals 0.6 with probability 10/11 and equals 0.001 with probability 1/11). However, by Jensen's inequality, if $v < 0$ then

$$1 > E\rho_0^{-1} = E(1/(1-\omega_0) - 1) \geqslant \frac{1 - E(1-\omega_0)}{E(1-\omega_0)} = \frac{1+\bar{v}}{1-\bar{v}},$$

and hence $v < 0$ implies that $\bar{v} < 0$. Thus, if the static speed $\bar{v}$ is positive, the RWRE may be transient to the left but if so, only with zero speed. We come back to this point in section 4.4.2, where we show that the last property is not necessarily true in high dimension.

(c) Another application of Jensen's inequality reveals that $|v| \leqslant |\bar{v}|$, with examples of strict inequality readily available, for example as in point (b) above. Thus, the random environment exhibits in general a *slowdown* with respect to the (averaged, deterministic) environment.

2.2. Limit laws, transient RWRE

Having clarified the ergodic behaviour of the RWRE, we turn to the discussion of limit laws in the transient setup, which turn out to be different under the annealed and quenched measures. We discuss in this section product measures P with $u := E(\log \rho_0) < 0$ (i.e., when the RWRE is transient to $+\infty$). Set $s = \sup\{r : E(\rho_0^r) < 1\}$ and note that because $u < 0$, necessarily $s \in (0, \infty]$. Set $T_i = \min\{n \geqslant 0 : X_n = i\}$ and $\tau_i = T_i - T_{i-1}$. The behaviour of the RWRE can be dramatically different from that of ordinary random walk, due to the existence of localized pockets of environments ('traps') where the walk spends a large time. We explain this point in some detail in section 3.3.

Theorem 2.3. *Suppose P is a uniformly elliptic i.i.d. environment with $u < 0$.*

(a) Suppose $s > 2$. Then there exists a deterministic constant $\sigma^2 > 0$ such that the sequence of random variables $W_n := (X_n - nv)/\sigma\sqrt{n}$ converges, under the annealed law, to a standard Gaussian random variable, that is

$$\mathbb{P}(W_n > x) \to_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y.$$

Further, with $Z_n(\omega) = v\sum_{i=1}^{[nv]}(E_\omega \tau_i - 1/v)$, and $\sigma_q^2 = |v|^3\left[\left[\mathbb{E}\left(\tau_1^2\right) - \mathbb{E}[(E_\omega\tau_1)^2]\right]\right]$, for P-almost every environment ω, the random variable $W_{n,q} := (X_n - nv - Z_n)/\sigma_q\sqrt{n}$ converges, under the quenched law P_ω^0, to a standard Gaussian random variable, that is

$$P_\omega^0(W_{n,q} > x) \to_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y, \qquad P - a.s.$$

(b) Suppose $s = 2$ and the law of $\log \rho_0$ is non-arithmetic. Then, for some deterministic constant σ, the random variable $(X_n - nv)/\sigma\sqrt{n \log n}$ converges, under the annealed law, to a standard Gaussian random variable.

(c) Suppose $s \in (1, 2)$ and the law of $\log \rho_0$ is non-arithmetic. Then, for some deterministic constant b, the random variable $(X_n - nv)/n^{1/s}$ converges, under the annealed law, to a stable random variable with parameters (s, b).

(d) Suppose $s = 1$ and the law of $\log \rho_0$ is non-arithmetic. Then, for some deterministic constants a, b, and some deterministic sequence $\delta(n)$ with $\delta(n) \sim an/\log n$, the random variable $(X_n - \delta(n))(\log n)^2/n^{1/s}$ converges, under the annealed law, to a stable random variable with parameters $(1, b)$.

(e) Suppose $s \in (0, 1)$ and the law of $\log \rho_0$ is non-arithmetic. Then, for some deterministic constant b, the random variable X_n/n^s converges, under the annealed law, to a stable random variable with parameters (s, b).

In the theorem above, a stable law with parameters (s, b) is the distribution of a random variable $\mathcal{S}$ with a characteristic function

$$E(\mathrm{e}^{\mathrm{i}t\mathcal{S}}) = \exp\left(-b|t|^s\left(1 + \mathrm{i}\frac{t}{|t|} f_s(t)\right)\right), \qquad t \in \mathbb{R},$$

where $f_s(t) = -\tan(\pi s/2)$ for $s \neq 1$ and $f_1(t) = (2/\pi)\log t$.

Remark 2.4

(a) Both the annealed and the quenched statements in part (a) carry over to a full invariance principle, which is convergence to a Brownian motion of the process $(X_{[nt]} - ntv)/\sigma\sqrt{n}$ (and $(X_{[nt]} - nvt - Z_{[nt]})/\sigma_q\sqrt{n}$) under the annealed law (respectively, quenched law). The annealed statement goes back to [KKS75]; see also [Zt04] for an extension to ergodic environment. The quenched statement is proved in [Zt04] with a weaker notion of convergence (convergence in probability). The version above is contained in [Pe07], and is valid for ergodic environments satisfying appropriate mixing conditions; see also [Gos06] for a similar result. It is worthwhile to note that the random centring $Z_n(\omega)$ is essential, and in fact the (annealed) variance of $Z_n(\omega)$ is of order n.

(b) The statements (b)–(e) are due to [KKS75], and are proved using an analysis of the hitting times τ_i. We refer to the regime described in this case as a *sub-diffusive* regime.

(c) When P is a strongly mixing environment, the parameter s has to be defined differently, by means of the large deviations rate function for the variable $n^{-1}\sum_{i=1}^{n}\log\rho_i$. An extension to part (a) for such environments is straight forward, we refer to [Zt04, Bre04a, Pe07] for several such extensions. Parts (b)–(e) are more delicate, and are not known for general ergodic environments with good mixing properties. For a class of Markovian environments, such a theorem holds, and the proof is contained in [MRZ04].

(d) There does not exist a quenched statement analogous to part (a) in the stable cases (b)–(e), and the actual limit law for hitting times can be shown to depend on the environment and on a specific subsequence of n's chosen. A study of this phenomenon is forthcoming in [Pe07].

2.3. *Limit laws and ageing, recurrent RWRE: Sinai's walk*

When $E(\log\rho_0) = 0$, the traps alluded to in section 2.2 stop being local, and the whole environment becomes a diffused trap. The walk spends most of its time 'at the bottom of the trap', and as time evolves it is harder and harder for the RWRE to move. This is the phenomenon of *ageing*, captured in the following theorem.

Theorem 2.5. *There exists a random variable B^n, depending on the environment only, such that for any $\eta > 0$,*

$$\mathbb{P}\left(\left|\frac{X_n}{(\log n)^2} - B^n\right| > \eta\right) \underset{n\to\infty}{\to} 0.$$

Further, for $h > 1$,

$$\lim_{\eta\to 0}\lim_{n\to\infty}\mathbb{P}\left(\frac{|X_{n^h} - X_n|}{(\log n)^2} < \eta\right) = \frac{1}{h^2}\left[\frac{5}{3} - \frac{2}{3}\,\mathrm{e}^{-(h-1)}\right]. \tag{2.3}$$

The first part of theorem 2.5 is due to Sinai [Si82], with Kesten [Ke86] providing the evaluation of the limiting law of B^n; see also [Go85]. It is actually not hard to understand the anomalous scaling $(\log n)^2$: indeed, the time for the particle to overcome a potential barrier of height $c_1\log n$ (refer to figure 1) is exponential in $c_1\log n$, i.e. an appropriate c_1 can be chosen such that this time is of order n. Hence, the range of the RWRE at time n cannot be larger than the distance in which the potential reached a height of $c_1\log n$. Due to the scaling properties of random walk, this distance is of order $(\log n)^2$.

The second part of theorem 2.5 is implicit in [Go85], and also follows from the analysis of the time spent by the RWRE at 'bottom of traps'. We refer to [LdMF99] for a detailed study of aging in the Sinai model by renormalization techniques, and to [DGuZ01] and [Zt04] for rigorous proofs that avoid renormalization arguments, and references. See also [Ch05] for a non-renormalization approach to some of the results in [LdMF99]. Finally, much information is available concerning the time spent by the walk at the most visited site (this time can be of order n in the Sinai model); see [Sh01, HuSh00, DGPS05] and [GaS02] for the transient case.

2.4. Tail estimates and large deviations

Another question of interest relates to the probability of seeing a-typical behaviour of the RWRE. These probabilities turn out to depend on the measure discussed, that is whether one considers the quenched or annealed measures.

Following Varadhan [V66], recall that a sequence of random variables $\mathcal{S}_n$ is said to satisfy the large deviations principle (LDP) with speed a_n and rate function I if, for any measurable set A with closure $\bar{A}$ and interior A^o,

$$-\inf_{x\in A^o} I(x) \leqslant \liminf_{n\to\infty} \frac{1}{a_n} \log P(\mathcal{S}_n \in A) \leqslant \limsup_{n\to\infty} \frac{1}{a_n} \log P(\mathcal{S}_n \in A) \leqslant -\inf_{x\in\bar{A}} I(x). \tag{2.4}$$

(Formally, the LDP holds if $P(\mathcal{S}_n \in A) \sim \mathrm{e}^{-nI(A)}$ where the equivalence is measured in an exponential scale and $I(A) = \inf_{x\in A} I(x)$.)

Cramér's theorem [DZ98, theorem 2.2.3] states that rescaled random walk S_n/n in a homogeneous environment with $\omega_i = \bar{\omega}$ for all i satisfies the LDP with speed n and strictly convex rate function $I(x)$ that vanishes only on $v_{\bar{\omega}} = 2\bar{\omega} - 1$. The situation is different for the RWRE.

Theorem 2.6. *The random variables X_n/n satisfy, for P-a.e. realization of the environment ω, a LDP under P_ω^0 with a deterministic convex rate function $I_P(\cdot)$. Under the annealed measure $\mathbb{P}$, they satisfy a LDP with convex rate function*

$$I(x) = \inf_{Q\in\mathcal{M}_1^e} (h(Q|P) + I_Q(x)), \tag{2.5}$$

where $h(Q|P)$ is the specific entropy of Q with respect to P and $\mathcal{M}_1^e$ denotes the space of stationary ergodic measures on Ω. Always, $I(x) \leqslant I_P(x)$, and both I and I_P may vanish for $x \in [0, v]$, and only for such x. In particular, neither I nor I_P need be strictly convex.

The rate function of the LDP for the RWRE thus differs from the case of homogeneous environments in two important aspects: it may vanish on the whole segment $[0, v]$, indicating sub-exponential behaviour for the probability of slowdown, and further the rate function is in general not strictly convex.

The quenched part of theorem 2.6 for i.i.d. environments is due to [GdH94]. We sketch in section 3.2 an argument that gives both the quenched and annealed LDP, and refer to [CGZ00, DGaZ04] for the general statements, proofs, and generalizations to non-i.i.d. environments. Note that theorem 2.6 means that to create an annealed large deviation, one may first 'modify' the environment (at a certain exponential cost measured by the specific entropy h) and then apply the quenched LDP in the new environment.

When $I(x)$ vanishes for $x \in [0, v]$, it means that the probability of seeing an a-typical slowdown of the random walk decays at a speed less than exponentially. Recall from theorem 2.1 that when P is a product measure with $E \log \rho_0 < 0$ and $s \in (1, \infty)$, X_n is transient to $+\infty$ with positive speed v, and necessarily also $P(\omega_0 < 1/2) > 0$, i.e. regions where the walk would tend to move in a direction opposite to v are possible.

Theorem 2.7 ([DPZ96, GZ98])**.** *Assume that P is a product measure with $E\rho_0 < 1$ and $s \in (1, \infty)$. Then, for any $w \in [0, v)$, $\eta > 0$, and $\delta > 0$ small enough,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(\frac{X_n}{n} \in (w-\delta, w+\delta)\right)}{\log n} = 1 - s, \tag{2.6}$$

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\eta}} \log P_\omega^0 \left(\frac{X_n}{n} \in (w-\delta, w+\delta)\right) = 0, \qquad P-a.s. \tag{2.7}$$

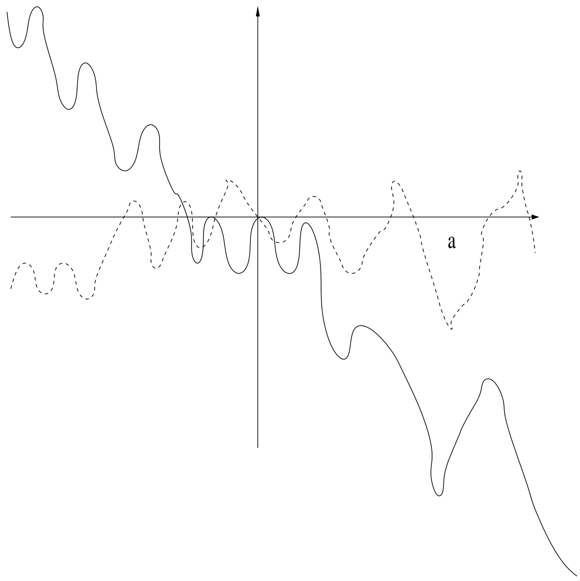

Figure 1. A typical realization of the potential $V(\cdot)$, in case $u < 0$ (solid) and $u = 0$ (dashed). In both cases, the environment tends to confine the particle near a.

and

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\eta}} \log P_\omega^0 \left(\frac{X_n}{n} \in (w-\delta, w+\delta) \right) = -\infty, \qquad P-a.s. \tag{2.8}$$

(Extensions of theorem 2.7 to the mixing environment setup are presented in [Zt04]. There are also precise asymptotics available in the case $s = \infty$ and $P(\omega_0 = 1/2) > 0$; see [PP99, PPZ99]).

Remark 2.8. One immediately notes the difference in scaling between the annealed and quenched slowdown estimates in theorem 2.7. These differences are due to the different nature of traps under the annealed and quenched measures; see sections 3.2 and 3.3.

3. One-dimensional RWRE: tools

In what follows, we introduce some of the tools involved in proving theorem 2.1, and provide additional information that can be obtained by using these tools.

3.1. Resistor networks

The transience and recurrence criterion in theorem 2.1 is proved by noting that conditioned on the environment ω, hitting probabilities for the Markov chain X_n can be directly related to properties of resistor networks [DoS84]. More explicitly, fix an interval $[-m_-, m_+]$ encircling the origin and for z in that interval, define

$$\mathcal{V}_{m_-,m_+,\omega}(z) := P_\omega^z(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+).$$

Define the resistance of a (non-oriented) edge $(i, i+1)$ by

$$\mathcal{R}_{(i,i+1)} := \begin{cases} \prod_{j=0}^{i} \rho_j =: \mathrm{e}^{V(i)}, & i \geqslant 0 \\ \prod_{j=1}^{-i-1} \rho_{-j}^{-1} =: \mathrm{e}^{V(i)}, & i < 0, \end{cases}$$

with the conductance $\mathcal{C}_{(i,i+1)} = \mathcal{R}_{(i,i+1)}^{-1}$. $V(\cdot)$ (see figure 1 for typical realizations) acts as a random potential for the motion of the RWRE, because the probability of jumping from i to $i+1$ can be checked to be precisely $\mathcal{C}_{(i,i+1)}/(\mathcal{C}_{(i-1,i)}+\mathcal{C}_{(i,i+1)})$. Then, for $z \in (m_-, m_+)$, $\mathcal{V}_{m_-,m_+,\omega}(z)$ equals the voltage at z across a resistor network with these conductances and voltage 1 at m_- and 0 and m_+, giving

$$\mathcal{V}_{m_-,m_+,\omega}(z) = \frac{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{z} \left(\prod_{j=i}^{z} \rho_j^{-1}\right)}. \tag{3.1}$$

The transience/recurrence criterion follows from (3.1), the ergodic theorem, and uniform ellipticity by noting for example that if $E \log \rho_0 < 0$ then

$$\limsup_{m_- \to \infty} \limsup_{m_+ \to \infty} \mathcal{V}_{m_-,m_+,\omega}(z) = 0.$$

We remark that the existence of a resistor network representation is equivalent to the model being *reversible*, a feature that will be lost in the case $d \geqslant 2$.

3.2. Recursions and hitting times

The proof of the LLN in theorem 2.1 is more instructive. Suppose $E \log \rho_0 \leqslant 0$ and recall the ($\mathbb{P}$-a.s. finite) hitting times $T_n = \min\{t \geqslant 0 : X_t = n\}$, and set $\tau_i = T_i - T_{i-1}$. Suppose that $\limsup_{n\to\infty} X_n/n = \infty$. One checks that τ_i is an ergodic sequence, hence $T_n/n \to \mathbb{E}(\tau_0)$, $\mathbb{P}$-a.s., which in turn implies that $X_n/n \to 1/\mathbb{E}(\tau_0)$, $\mathbb{P}$-a.s. But,

$$\tau_0 = \mathbf{1}_{\{X_1=1\}} + \mathbf{1}_{\{X_1=-1\}}(1 + \tau'_{-1} + \tau'_0),$$

where τ'_{-1} (respectively, τ'_0) denote the first hitting time of 0 (respectively, 1) for the random walk X_n after it hits -1. Hence, taking P_ω^0 expectations, and noting that $\left\{E_\omega^i(\tau_i)\right\}_i$ are, P-a.s., either all finite or all infinite,

$$E_\omega^0(\tau_0) = \frac{1}{\omega_0} + \rho_0 E_\omega^{-1}(\tau_{-1}). \tag{3.2}$$

When P is a product measure, ρ_0 and $E_\omega^{-1}(\tau_{-1})$ are P-independent, and taking expectations results with $\mathbb{E}(\tau_0) = (1+E(\rho_0))/(1-E(\rho_0))$ if the right-hand side is positive and ∞ otherwise, from which (2.2) follows. The ergodic case is obtained by iterating relation (3.2).

The hitting times T_n are also the beginning of the study of limit laws for X_n. To appreciate this in the case of product measures P with $E(\log \rho_0) < 0$ (i.e., when the RWRE is transient to $+\infty$), one first observes, after some computations, that from the above recursions,

$$\mathbb{E}\left(\tau_0^r\right) < \infty \Longleftrightarrow E\left(\rho_0^r\right) < 1. \tag{3.3}$$

We emphasize that quenched, the random variables τ_i are independent but *not* identically distributed. Annealed, they form a stationary (but not independent) sequence, and, with $s = \sup\left\{r : E\left(\rho_0^r\right) < 1\right\}$, they possess all moments up to (but not including) s. When $s > 2$, this and the Lindeberg–Feller criterion for the validity of the CLT, lead to the proof of part (a) of theorem 2.3. Parts (b)–(e) in that theorem are more delicate, since to prove convergence to a stable distribution one needs a good control on tails and in particular a regular-varying condition on the tails of the summands. Recursions again play a key role there, but we do not discuss further details here.

We finally note that recursions involving the hitting times T_n are also crucial when proving the quenched LDP in theorem 2.6. Indeed, standard large deviations arguments for which we refer to [DZ98] show that in order to prove the quenched LDP for T_n/n, it is enough to understand the behaviour of

$$\lim_{n\to\infty}\frac{1}{n}\log E_\omega^0(\mathrm{e}^{\lambda T_n}) = \lim_{n\to\infty}\frac{\sum_{i=1}^n \log E_\omega^0(\mathrm{e}^{\lambda\tau_i})}{n} = E\log E_\omega^0(\mathrm{e}^{\lambda\tau_1}) =: \Lambda(\lambda),$$

where the second equality is due to the ergodic theorem. But,

$$E_\omega^0(\mathrm{e}^{\lambda\tau_1}) = \omega_0\,\mathrm{e}^{\lambda} + (1-\omega_0)\,\mathrm{e}^{\lambda}E_\omega^{-1}(\mathrm{e}^{\lambda\tau_0})E_\omega^0(\mathrm{e}^{\lambda\tau_1}),$$

where τ_0 denotes the time that a random walk, started at -1, hits 0. Iterating, one gets

$$E_\omega^0(\mathrm{e}^{\lambda\tau_1}) = \frac{\omega_0}{\mathrm{e}^{-\lambda} - (1-\omega_0)\frac{\omega_{-1}}{\mathrm{e}^{-\lambda}-(1-\omega_{-1})\cdots}},$$

leading to an expression of $\Lambda(\lambda)$ as the expectation of the logarithm of this continued fraction, and to $I_P(x) = \sup_\lambda[\lambda x - \Lambda(\lambda)]$ being the Legendre transform of Λ (technical details, involving for example proving that the critical value of λ for which the continued fraction converges is deterministic, are omitted in this discussion and can be found in [CGZ00]). The annealed statement is an application of Varadhan's lemma [DZ98, theorem 4.3.1] of large deviations theory (a.k.a. Laplace's method): we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}(\mathrm{e}^{\lambda T_n}) &= \lim_{n\to\infty}\frac{1}{n}\log E\left(\mathrm{e}^{n\cdot\frac{1}{n}\log E_\omega^0(\mathrm{e}^{\lambda T_n})}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\log E\left(\mathrm{e}^{n\cdot\frac{1}{n}\sum_{i=1}^n\log E_\omega^0(\mathrm{e}^{\lambda\tau_i})}\right)\\ &=: \lim_{n\to\infty}\frac{1}{n}\log E\left(\mathrm{e}^{\sum_{i=1}^n G_i(\omega)}\right),\end{aligned}$$

where

$$G_i(\omega) = \log E_\omega^{i-1}\,\mathrm{e}^{\lambda\tau_i} = \log E_{\theta^{i-1}\omega}^0\,\mathrm{e}^{\lambda\tau_1},$$

and $\theta^i\omega$ denotes an i-shift of the environment ω, i.e. $(\theta^i\omega)_0 = \omega_i$. Since the empirical measures $n^{-1}\sum_{i=1}^n\delta_{\theta^i\omega}$ satisfy the LDP with speed n and rate function equaling the specific entropy $h(\cdot|P)$, see [DnV83], [DZ98, chapter 6], the annealed LDP and formula (2.5) follow, after one takes care of the (non-negligible) technicalities.

3.3. Traps and slowdown estimates

As already mentioned, the unusual behaviour of one-dimensional RWRE, and in particular of the various slowdown regimes in theorem 2.7, is best understood in terms of the existence of traps in the environment, which are due to barriers in the potential $V(\cdot)$. To demonstrate the role of traps, let us exhibit, for $w = 0$, a lower bound that captures the correct behaviour in the annealed setup, and that forms the basis for the proof of the more general statement. Indeed, $\{X_n \leqslant \delta\} \subset \{T_{n\delta} \geqslant n\}$. Fixing $R_k = R_k(\omega) := k^{-1}\sum_{i=1}^k \log\rho_i$, it holds that R_k satisfies a large deviation principle with rate function $J(y) = \sup_\lambda\left(\lambda y - \log E\left(\rho_0^\lambda\right)\right)$, and it is not hard to check that $s = \min_{y\geqslant 0} y^{-1}J(y)$. Fixing a y such that $J(y)/y \leqslant s+\eta$, and $k = \log n/y$, one checks that the probability that there exists in $[0, \delta n]$ a point a with $R_k\circ\theta^a\omega \geqslant y$ is at least $n^{1-s-\eta}$ (such a point will serve as a potential barrier, like the point a in figure 1). But, the probability that the RWRE does not cross such a segment by time n is, due to (3.1), bounded away from 0 uniformly in n. This yields the claimed lower bound in the annealed case. In the quenched case, one has to work with traps of size almost $k = \log n/sy$ for which $kR_k \geqslant y$, which occur with probability 1 eventually, and use (3.1) to compute the probability of an atypical slowdown inside such a trap. The fluctuations in the length of these typical traps is the reason why the slowdown probability is believed, for P-a.e. ω, to fluctuate with n, in the sense that

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^0\left(\frac{X_n}{n} \in (-\delta,\delta)\right) = -\infty, \qquad P-a.s., \tag{3.4}$$

while it is known that

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^0\left(\frac{X_n}{n} \in (-\delta,\delta)\right) = 0, \qquad P-a.s.$$

The limit (3.4) has been demonstrated rigorously in some particular cases, see [Ga02], but the general case, which is stated as a conjecture in [GZ98], is still open.

3.4. Homogenization and the environment viewed from the point of view of the particle

We have neither discussed nor used so far a standard approach in the study of motion in random media, namely homogenization via the study of the environment viewed from the point of view of the particle. We now discuss this approach in the context of random walks in random environments in dimension $d = 1$, where it leads to alternative proofs of the LLN and (annealed) CLT, when $v > 0$. A detailed exposition appears in [Ko85, Mo94, Sz04] and [Zt04].

As above, we let $\theta : \Omega \to \Omega$ denote the spatial shift acting on the environment. Set $\bar\omega(n) = \theta^{X_n}\omega$. It is immediate to check that $\bar\omega_n$ is a Markov chain with state space Ω. Suppose now that P is ergodic and $v > 0$, and define the probability measure Q on ω by the equality, valid for any measurable $B \subset \Omega$,

$$Q(B) = E_{\mathbb{P}}\left(\sum_{i=1}^{T_1-1} \mathbf{1}_{\bar\omega(i)\in B}\right), \qquad \bar Q(B) = \frac{Q(B)}{Q(\omega)} = \frac{Q(B)}{E_{\mathbb{P}}(T_1)}.$$

One then checks that $\bar Q$ is an invariant measure for the Markov chain $\bar\omega(n)$, and that $\mathrm{d}\bar Q/\mathrm{d}P \in (0,\infty)$. Further, due to uniform ellipticity, $\bar Q$ is actually ergodic, and hence the ergodic theorem implies that $u_n := n^{-1}\sum_{i=1}^n (2\bar\omega(i)_0 - 1)$ converges to a deterministic limit $\bar v$ for $\bar Q$ almost every initial condition $\bar\omega(0) = \omega$, and hence, for P almost every such initial condition. Since $M_n := X_n - nu_n$ is a martingale with bounded increments, it follows that $X_n/n \to \bar v$, P-almost surely, and hence $\bar v = v$ and the LLN in theorem 2.1 for $s > 1$ follows.

Standard Martingale arguments also show that $M_n/\sqrt{n}$ satisfies the CLT, however it is not easy to deduce from this a CLT for $(X_n - nv)/\sqrt{n}$ due to the fluctuations of $\sqrt{n}u_n$. Instead, the homogenization proof of the (annealed) CLT, for i.i.d. and certain mixing environments, involves the construction of a *corrector*, or harmonic coordinates. This proceeds as follows. One seeks a function $h(x,\omega)$ such that $M_n = X_n - nv + h(X_n,\omega)$ is a martingale (with respect to the natural filtration of (X_n), and the measure P_ω^0). Such an h can be computed, and in fact, with $\Delta(x,\omega) = h(x+1,\omega) - h(x,\omega)$, it holds that Δ satisfies the equation

$$\Delta(x,\omega) = -\frac{2\omega_x - 1 - v}{\omega_x} + \rho_x \Delta(x-1,\omega), \tag{3.5}$$

which can be solved explicitly. The normalized increasing process $n^{-1}\sum_{i=1}^n E((M_{i+1} - M_i)^2 | X_j, j \leqslant i)$ converges by the same argument that gave the LLN. Therefore, $M_n/\sqrt{n}$ satisfies the CLT under the quenched measure P_ω^0, with a deterministic limiting variance σ_1^2, and an additional argument shows that $|h(X_n, \omega) - h(nv, \omega)| \to 0$ when $s > 2$ and mixing conditions are satisfied by the environment, $h(nv, \omega)/\sqrt{n}$ converges in distribution, under P, to a Gaussian variable (which, since σ_1^2 is deterministic under the quenched measure P_ω^0, is asymptotically independent of $M_n/\sqrt{n}$). Combining these two facts yields the (annealed) CLT for $(X_n - nv)/\sqrt{n}$, i.e. the annealed statement in part (a) of theorem 2.3.

4. Multi-dimensional RWRE–non-perturbative regime

We turn our attention to RWRE in the lattice $\mathbb{Z}^d$ with $d > 1$. Unless stated otherwise explicitly, we only consider in the following measures P that are i.i.d. and uniformly elliptic. While the case of $d = 1$ serves as motivation, the lack of reversibility means that there is no natural analogue of the random potential $V(\cdot)$.

4.1. Ergodic properties and a $0 - 1$ *law*

A natural starting point for the discussion of ergodic properties of the RWRE (X_n) would have been an analogue of theorem 2.1. Unfortunately, obtaining such a statement has been a major challenge since the early 1980s, and is still open. To explain the challenge, we need to digress and introduce a certain conjectured $0 - 1$ law.

Fix $\ell \in S^{d-1}$, i.e. ℓ is a unit vector in $\mathbb{R}^d$. Define the events

$$A_\ell^+ = \{\lim_{n\to\infty} X_n \cdot \ell = \infty\}, \qquad A_\ell^- = \{\lim_{n\to\infty} X_n \cdot \ell = -\infty\}.$$

The proof of the following proposition, due to Kalikow [Ka81], is easy and is sketched in section 5.1.

Proposition 4.1. *Assume P is i.i.d. and elliptic, i.e. $P(\omega(0, e) > 0) = 1$ for all e with $|e| = 1$, and that $\ell \in S^{d-1}$. Then, $\mathbb{P}(A_\ell^+ \cup A_\ell^-) \in \{0, 1\}$.*

Note that for $d = 1$, theorem 2.1 implies that $\mathbb{P}(A_\ell) \in \{0, 1\}$. If one ever hopes to obtain a LLN, then one should be able to prove the following.

Conjecture 4.2 (Kalikow)**.** *Assume P is i.i.d. and uniformly elliptic, and that $\ell \in S^{d-1}$. Then, $\mathbb{P}(A_\ell^+) \in \{0, 1\}$.*

Efforts to prove conjecture 4.2 are ongoing. The following summarizes its status at the current time.

Theorem 4.3

(a) Conjecture 4.2 holds true for $d = 1, 2$ and elliptic i.i.d. environments.
(b) There exist ergodic environments that are elliptic (for $d = 2$) and even uniformly elliptic and mixing (for $d \geqslant 3$), for which a deterministic direction $\ell \in S^{d-1}$ exists such that $P_\omega^0(A_\ell) \in (0, 1)$, for P-almost every ω.

As mentioned above, part (a) of theorem 4.3 for $d = 1$ is a direct consequence of the LLN, theorem 2.1. Parts (a) and (b) of theorem 4.3 for $d = 2$ are due to [ZrM01], while part (b) for $d \geqslant 3$ is due to [BrZZ06]. We provide a sketch of the proofs in section 5.1.2.

As it turns out, the validity of conjecture 4.2 is the only obstruction to a LLN. In fact, the following holds.

Theorem 4.4. *Assume P is i.i.d. and uniformly elliptic.*

(a) Fix $\ell \in S^{d-1}$. Then,

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_+ \mathbf{1}_{A_\ell^+} + v_- \mathbf{1}_{A_\ell^-}, \qquad \mathbb{P} - a.s.. \tag{4.1}$$

In particular, when $d = 2$ the LLN holds true.

(b) P-almost surely, there are at most two possible limit points, denoted by v_1, v_2, for the sequence X_n/n. Further, v_1, v_2 are deterministic, and if $v_1 \neq v_2$ then there exists a constant $a \geqslant 0$ such that $v_2 = -a v_1$.

(c) When $d \geqslant 5$, if $v_1 \neq v_2$ then at least one of v_1 and v_2 equals 0.

Part (a) of theorem 4.4 is due to [SzZr99] and [Zr02]. Part (b) is proved explicitly in [V04] and [Be06]. Part (c) is due to [Be06]. Of course, part (a) of the theorem implies that if conjecture 4.2 is true, then the LLN holds for P i.i.d. and uniformly elliptic.

The proof of theorem 4.4, and of many of the other results in this section, uses the machinery of regeneration times, introduced in [SzZr99]. Roughly, a random time k is a regeneration time relative to a direction $\ell \in S^{d-1}$ if $X_k \cdot \ell \geqslant X_n \cdot \ell$ for all $k \geqslant n$ but $X_k \cdot \ell < X_n \cdot \ell$ for all $k < n$ (i.e., $X_n \cdot \ell$ sets a record at time k, and never moves backward from that record). It will turn out that the sequence of inter-regeneration times and inter-regeneration distances is an i.i.d. sequence under the annealed measure $\mathbb{P}$, if P is i.i.d.; see lemma 5.1. Once such an i.i.d. sequence has been identified, ergodic arguments yield the LLN, and the CLT involves studying tail behaviour of the regeneration times. We provide further details in section 5.1.4.

We note that so far, there is no known criterion that allows one to decide the question of transience or recurrence for RWRE in dimension $d \geqslant 2$, although one certainly expect transience as soon as $d \geqslant 3$, for i.i.d. uniformally elliptic environments.

4.2. Ballistic behaviour and Sznitman's conditions

Lacking an explicit expression for the speed of the RWRE for $d \geqslant 2$, a natural goal is to identify a large family of models for which $X_n/n \to v \neq 0$. RWREs that satisfy such a relation are called *ballistic*. As we saw in theorem 2.1, when $d = 1$ and $X_n \to \infty$, and the environment is i.i.d., the RWRE is ballistic if and only if $E\rho_0 < 1$.

Define $d_0 := \sum[\omega(0, e_i) - \omega(0, -e_i)]e_i$ as the drift at the origin. Of course, if there exists a direction $\ell \in S^{d-1}$ such that $d_0 \cdot \ell > 0$ for P-a.e. environment, a simple martingale argument shows that $X_n/n \to v$ with $v \cdot \ell > 0$. However, as we show below (see remark 4.15), one should not confuse the condition $Ed_0 \neq 0$ with ballistic behaviour, as it neither guarantees nor is sufficient to ensure a limiting nonzero speed.

Following Zerner [Zr98], we call such environments for which there exists a deterministic ℓ such that $d_0 \cdot \ell > 0$, P-a.s., *non-nestling*. We will be mainly interested in *nestling* environments, that is environments in which the origin belongs to the closed convex hull of the support of d_0 (the source for the name lies in the fact that when the walk is nestling, it is possible to construct localized regions where the walk return many times, leading to the mental picture of a bird that keeps returning to a nest). Such regions can serve as traps and slow down the particle. However, unlike $d = 1$, all attempts to build explicit traps that slow down the particle to a sub-diffusive scale quickly fail. One thus suspects that a good control of trapping properties will lead to an analysis of the RWRE.

With this motivation in mind, we follow Sznitman in introducing some condition on the environment that will eventually lead to a good understanding of the ballistic regime. Fix a direction $\ell \in S^{d-1}$, and for $b > 0$, define the region $U_{\ell,b,L} = \{x \in \mathbb{Z}^d : x \cdot \ell \in (-bL, L)\}$. Let $T_{\ell,b,L} = \min\{n > 0 : X_n \notin U_{\ell,b,L}\}$.

Definition 4.5. *Let $\gamma \in (0, 1)$ be given. Then, P satisfies condition T_γ relative to ℓ if for all ℓ' in some neighbourhood of ℓ, and all $b > 0$,*

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log \mathbb{P}\big(X_{T_{\ell,b,L}} \cdot \ell < 0\big) < 0. \tag{4.2}$$

P satisfies condition T' relative to ℓ if it satisfies condition T_γ relative to ℓ for all $\gamma \in (0, 1)$. It satisfies condition T relative to ℓ if it satisfies condition T_1 relative to ℓ.

In words, condition T relative to ℓ holds if the exit from a slab that is contained between two hyperplanes perpendicular to ℓ, located respectively at distance $+L$ in the ℓ direction and $-bL$ in the opposite direction, occurs through the 'backward' direction with probability that is exponentially small in L. Condition T' relaxes the exponential decay to 'almost' exponential decay (there is an alternative description of condition T' in terms of regeneration distances; see proposition 5.3). The power of condition T' is the following.

Theorem 4.6 (Sznitman)**.** *Assume P is i.i.d. and uniformly elliptic, and that condition T' relative to some direction ℓ holds. Then, the process (X_n) is ballistic, i.e. $X_n/n \to v \neq 0$ for some deterministic v with $v \cdot \ell > 0$, and there is a deterministic $\sigma^2 > 0$ such that, under the annealed measure $\mathbb{P}$, $(X_n - nv)/\sigma\sqrt{n}$ converges in distribution to a standard Gaussian random variable.*

(The convergence in distribution in theorem 4.6 actually extends to an invariance principle.)

A simple martingale argument implies that condition T (and hence T') holds for a certain direction ℓ when the environment is non-nestling. We next describe sufficient conditions that imply condition T' for certain nestling environments.

4.2.1. Kalikow's approach. Fix a strict finite subset U of $\mathbb{Z}^d$ that contains the origin, and let τ_U denote the first exit time from U. Define an auxiliary Markov chain on U and its boundary by

$$P_U(x, x+e) = \frac{\mathbb{E}\big[\sum_{n=0}^{\tau_U} \mathbf{1}_{X_n=x}\,\omega(x, x+e)\big]}{\mathbb{E}\big[\sum_{n=0}^{\tau_U} \mathbf{1}_{X_n=x}\big]}, \qquad x \in U,$$

with the walk stopped when exiting U (P_U is a transition kernel which weights the law of $\omega(x, x+e)$ according to the number of visits to x before time τ_U). It is an easy exercise to check that the exit law from U are the same under the Markov chain generated by P_U and under the annealed measure $\mathbb{P}$. In view of that, the following condition may be natural.

Definition 4.7 (Kalikow)**.** *We say that Kalikow's condition relative to ℓ is satisfied if*

$$\epsilon_\ell := \inf_{U, x\in U} \sum_{|e|=1} P_U(x, x+e)(\ell \cdot e) > 0. \tag{4.3}$$

Theorem 4.8

(a) Assume P is i.i.d. and uniformly elliptic, with ellipticity constant κ. Assume Kalikow's condition relative to ℓ holds. Then, so does condition T relative to ℓ.
(b) Assume $E((d_0 \cdot \ell)_+) \geqslant \kappa^{-1} E((d_0 \cdot \ell)_-)$. Then, Kalikow's condition relative to ℓ holds.

Kalikow's condition was introduced by him in [Ka81] as a way to prove the $0 - 1$ law for a (nontrivial) class of examples. For $d = 1$, it is easy to check that it is equivalent to ballistic behaviour, i.e. to $s > 1$. Kalikow's condition was used in [SzZr99] in order to analyse regeneration times and prove a LLN, and in [Sz00] in order to prove a CLT. It is an easy martingale argument to verify that it implies condition T relative to ℓ.

4.2.2. Sznitman's effective criterion for condition T'. The verification of condition T' seems *a priori* not obvious. It is thus extraordinary that an effective criterion for checking it exists.

Let $\ell \in S^{d-1}$ be given. Let $\mathcal{O} : \mathbb{R}^d \to \mathbb{R}^d$ denote a rotation with $\mathcal{O}e_1 = \ell$. Let $B = \mathcal{O}((-(L-2), L+2) \times (-\tilde{L}, \tilde{L})^{d-1})$ denote a box with sides $2L+4$ (in the ℓ direction) and $2\tilde{L}$ (in all other directions), symmetric with respect to reflections around 0. Let $\partial_+ B$ denote the part of the boundary of B consisting of points x with $x \cdot \ell \geqslant L+2$ and $|\mathcal{O}e_i \cdot x| \leqslant \tilde{L}$, for all $i \geqslant 2$ ($\partial_+ B$ consists of those points that are on the part of the boundary that belongs to the hyperplane that is both perpendicular to ℓ and has positive ℓ displacement). Finally, let $\rho_B = \rho_B(\omega) := P^0_\omega\big(X_{T_B} \notin \partial_+ B\big) \big/ P^0_\omega\big(X_{T_B} \in \partial_+ B\big)$.

Theorem 4.9. *There exist constants $c_1 = c_1(d), c_2 = c_2(d) > 1$ such that if P is i.i.d. and uniformly elliptic, and $\ell \in S^{d-1}$, then condition T' relative to ℓ is equivalent to*

$$\inf_{\substack{B\in\mathcal{B}_{c_1,c_2}\\ 0<\alpha\leqslant 1}} \left\{c_1 |\log \kappa|^{3(d-1)} (\tilde{L})^{d-1} L^{3(d-1)+1} E\big(\rho_B^\alpha\big)\right\} < 1,$$

where $\mathcal{B}_{c_1,c_2}$ denotes the collections of boxes B as above with $L \geqslant c_2$ and $\tilde{L} \in [3\sqrt{d}, L^3]$.

Theorem 4.9 appears in [Sz02], and is used in [Sz03a] to construct an example of a ballistic RWRE that does not satisfy Kalikow's condition but does satisfy T', relative to some ℓ. It is also useful when discussing environments that are small perturbations of simple random walk; see section 6.1.

4.2.3. Sznitman's conjecture. As we discuss in proposition 5.3, condition T' is equivalent to certain exponential moments on the maximal distance from the origin the RWRE has achieved before time τ_1. In the course of proving theorem 4.9, Sznitman actually proves that condition T_γ relative to ℓ with any $\gamma \in (1/2, 1)$ implies condition T' relative to the same ℓ. This led him to the following conjecture; see [Sz02]:

Conjecture 4.10 (Sznitman)**.** *Assume P is uniformly elliptic and i.i.d. Then, condition T relative to ℓ is implied by condition T_γ relative to ℓ for any $\gamma \in (0, 1)$.*

It is also reasonable to expect ('plausible', in the language of [Sz02]) that in addition, ballistic behaviour with speed v implies condition T relative to $\ell = v/|v|$, for $d > 1$.

For $d = 1$, and i.i.d. environment, all the conditions T_γ with respect to the direction $\ell = 1$ are equivalent to $E \log \rho_0 < 0$; see [Sz99]. Hence, conjecture 4.10 holds when $d = 1$ (note that this is *not* the case for the conclusions concerning ballistic behaviour, which do not hold true for $d = 1$).

We note in passing that in the ballistic situation, some information on the environment viewed from the point of view of the particle can be deduced. We refer to [BoS02] and [RA03] for details.

4.3. Large deviations, quenched and annealed

In dimension $d = 1$, the large deviations for the sequence X_n/n were obtained by considering hitting times. While this approach can be partially extended to obtain quenched LDPs for some RWREs (see [Zr98]), its scope is limited, and in particular it does not apply to all i.i.d. measures P, nor to an annealed LDP.

A different approach was taken by Varadhan [V04], who obtained the following.

Theorem 4.11. *Assume $d \geqslant 2$.*

(a) Assume P is a uniformly elliptic, ergodic measure. Then, for P-a.e. environment ω, the sequence of variables X_n/n under P_ω^0 satisfies the (quenched) LDP with speed n and deterministic, convex rate function I.

(b) Assume further that P is i.i.d. Then, the sequence of random variables X_n/n satisfies, under $\mathbb{P}$, the (annealed) LDP with speed n and convex rate function $\mathcal{I}$.

(c) The rate functions I and $\mathcal{I}$ possess the same zero set. Further, this (convex) set is either a single point or a segment of a line.

As for dimension $d = 1$, both I and $\mathcal{I}$ are in general not strictly convex.

The quenched statement (part (a)) is an application of the ergodic sub-additive theorem [Li85], noting that

$$\begin{aligned} P_\omega^0(X_{n+m} = [(n+m)u]) &\geqslant P_\omega^0(X_n = [nu])P_\omega^{[nu]}(X_m = [(n+m)u]) \\ &= P_\omega^0(X_n = [nu])P_{\theta^{[nu]}\omega}^0(X_m = [(n+m)u] - [nu]), \end{aligned}$$

which together with the ellipticity that is used to smooth the integer effects above, leads to the quenched LDP. The annealed LDP is obtained by noting that the process of histories of the walk is a Markov chain, and applying the general large deviations theory for such chains. The details are rather involved and we do not bring them here, referring the reader to [V04].

Remark 4.12

(a) In the multi-dimensional case, a formula like (2.5), with its intuitive description of the way an annealed deviation is obtained, is not available, since the modification of big chunks of the environment has probability which decays exponentially in volume order, instead of n.

(b) An alternative description of the quenched rate function, that is more instructive than the sub-additivity argument, has been developed for the related model of diffusions in random environments in [KRV06].

(c) Part (b) of theorem 4.11 was extended to certain mixing environments in [RA04].

As for $d = 1$, it is natural to study slowdown estimates in the region where the rate functions vanish, and in particular to study the probability of slowdown. This study is closely related to the analysis of condition T'. For nestling environments, it is easy to exhibit a lower bound, based on traps as in dimension 1, that shows that the slowdown probability decays slower than exponentially in n, and the challenge is to prove matching upper bounds. For P satisfying Kalikow's condition, the best currently available results are in [Sz99] and [Sz00].

4.4. Non-ballistic results

The analysis of RWRE for environments that do not exhibit ballistic behaviour is still limited. Still, two important classes of models have been identified, for which the analysis could be carried out. We sketch those below.

4.4.1. Balanced environment. A particular class of environments worth mentioning is the class of *balanced* environments, where $\omega(0, e_i) = \omega(0, -e_i)$ for all i, in which case $d_0 = 0$. In that case, X_n itself is a martingale with bounded increments, and thus $X_n/n \to 0$, $\mathbb{P}$-a.s. Much more can be said.

Theorem 4.13. *Assume P is stationary and ergodic, balanced and uniformly elliptic. Then $X_n/n \to 0, \mathbb{P}$-a.s., and there exists a deterministic $\sigma^2 > 0$ such that $X_n/\sigma\sqrt{n}$ converges in* *distribution (under the annealed measure* $\mathbb{P}$*) to a Gaussian random variables. Further,* X_n *is recurrent if* $d = 2$ *and transient if* $d \geqslant 3$*.*

Theorem 4.13 is essentially due to [L85] (the recurrence statement is due to Kesten, and can be found in [Zt04]). It is one of the few instances where 'classical' homogenization can be applied to the study of multi-dimensional RWRE. See section 5.2 for some further details.

4.4.2. RWRE with deterministic components. A key to the analysis of the ballistic case is the existence of certain regeneration times. Those were used to create an i.i.d. sequence under the measure $\mathbb{P}$.

In the non-ballistic case, regeneration times as defined above do not exist. However, if the dimension of the space is large enough and some of the components are deterministic, an alternative to regeneration times can be found, based on *cut times* for simple random walk. We postpone to section 5.3 the precise definition of cut times and the sketch of the proof of the following result, which is due to [BoSZ03].

Theorem 4.14. *Assume* $d = d_1 + d_2$ *with* $d_1 \geqslant 5$*. Assume* P *is a uniformly elliptic i.i.d. measure, with* $\omega(x, x + e) = q(e)$ *for* $e = \pm e_i, i = 1, \ldots, d_1$ *and a deterministic* q*. Then, there exists a deterministic constant* v *such that* $X_n/n \to v$*,* $\mathbb{P}$*-a.s. Further, if* $d_1 \geqslant 13$*, then the quenched CLT holds, i.e. there exists a deterministic* $\sigma^2 > 0$ *such that* $(X_n - nv)/\sigma\sqrt{n}$ *converges in distribution to a standard Gaussian variable.*

Remark 4.15

(a) The convergence in distribution in theorem 4.14 extends to a full invariance principle.
(b) An amusing consequence of theorem 4.14, is that for $d > 5$, one may construct P i.i.d. and uniformly elliptic such that $E(d_0 \cdot \ell) < 0$ but the resulting RWRE is ballistic with $v \cdot \ell > 0$. Recall that this is impossible in dimension $d = 1$; see remark 2.2(b). Also, for $d > 6$, one may construct for every $\epsilon > 0$ a P i.i.d. and uniformly elliptic such that $|\omega(x, x + e) - 1/2d| < \epsilon$, $E(d_0) \neq 0$, but $X_n/n \to 0$, $\mathbb{P}$-a.s., or such that $E(d_0) = 0$ but the walk is ballistic. We refer to [BoSZ03] for the construction.

5. Multi-dimensional RWRE: non-perturbative tools

We present in this section those tools that are used in proving the results in section 4. Unless otherwise stated, we assume that $\mathbb{P}$ is i.i.d. and uniformly elliptic.

5.1. Regeneration times

5.1.1. Definitions and a proof of proposition 4.1. We begin by introducing the notion of regeneration times with respect to a direction ℓ. In what follows, we let $Z_n = X_n \cdot \ell$. Call a time k *fresh relative to* ℓ if $Z_k > Z_n, \forall n < k$. A fresh time k is called a *regeneration time relative to* ℓ if $Z_n \geqslant Z_k$ for all $n \geqslant k$. The sequence of regeneration times is denoted by τ_i; see figure 2 for an illustration of the definition of regeneration times.

We can now provide a sketch of the proof of proposition 4.1: assume that $\mathbb{P}(A_\ell^+) > 0$. Then, by the Markov property and stationarity of the environment, each fresh time relative to ℓ has a uniformly bounded away from zero $\mathbb{P}$-probability to be a regeneration time relative to ℓ . Thus, if $\mathbb{P}(A_\ell^+) > 0$ then the existence of infinitely many fresh times relative to ℓ implies that A_ℓ^+ occurs. On the other hand, if $\mathbb{P}(A_\ell^+) = 0$ then for every $z \in \mathbb{Z}^d$ with $z \cdot \ell > -K$, and every K,

$$P_\omega^z(\text{there exists } n > 0 \text{ with } Z_n < -K) = 1, \qquad \mathbb{P} - a.s.$$

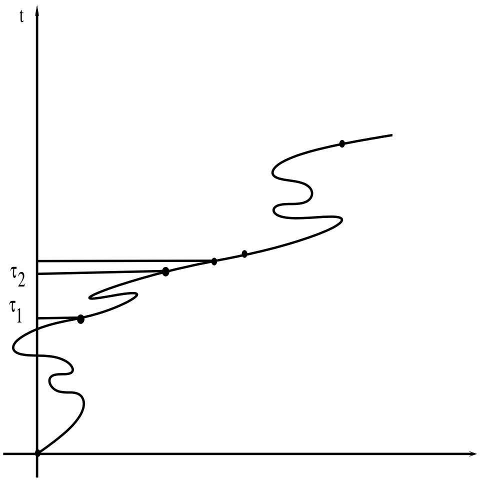

Figure 2. The projection of the path $X_t \cdot \ell$ (horizontal axis) versus time t (vertical axis), and the first two regeneration times τ_1 and τ_2.

Similarly, if $\mathbb{P}(A_\ell^-) > 0$ then the existence of infinitely many fresh times relative to $-\ell$ implies that A_ℓ^- occurs, while $\mathbb{P}(A_\ell^-) = 0$ implies that for every $z \in \mathbb{Z}^d$ with $z \cdot \ell < K$, and every K,

$$P_\omega^z(\text{there exists } n > 0 \text{ with } Z_n > K) = 1, \qquad \mathbb{P} - a.s.$$

Therefore, if $\mathbb{P}(A_\ell^-) = 0$ then there will be infinitely many fresh times relative to ℓ, implying that if $\mathbb{P}(A_\ell^+) > 0$ then $\mathbb{P}(A_\ell^+) = 1$. Hence, $\mathbb{P}(A_\ell^-) = 0$ implies $\mathbb{P}(A_\ell^+) \in \{0, 1\}$, and consequently $\mathbb{P}(A_\ell^+ \cup A_\ell^-) \in \{0, 1\}$ in this case. The same argument applies to $P(A_\ell^+) = 0$. Finally, if $P(A_\ell^+)P(A_\ell^-) > 0$, then either there are infinitely many fresh points relative to ℓ (and consequently, A_ℓ^+ occurs), or there are infinitely many fresh points relative to $-\ell$ (and consequently, A_ℓ^- occurs). Therefore, $P(A_\ell^+)P(A_\ell^-) > 0$ implies $\mathbb{P}(A_\ell^+ \cup A_\ell^-) = 1$.

5.1.2. Independence properties and the LLN. The most important feature of regeneration times is their independence properties. In what follows, we fix a direction ℓ, and say simply that k is a regeneration time if it is a regeneration time relative to ℓ. Let $\tau_i, i \geqslant 1$ denote the sequence of regeneration times (a consequence of the argument above is that if $\tau_1 < \infty$ then, $\mathbb{P}$-a.s., there are infinitely many regeneration times). Let $D = \min\{n > 0 : X_n \cdot \ell < 0\}$. The following lemma describes the independence properties of regeneration times.

Lemma 5.1. *Assume $\mathbb{P}$ is i.i.d., with $\mathbb{P}(A_\ell) > 0$. Then, the following hold.*

(a) The sequence of random vectors

$$\mathcal{V}_i := \left\{(\tau_{i+1} - \tau_i), \left(X_{n+\tau_i} - X_{\tau_i}\right)_{0 \leqslant n \leqslant \tau_{i+1}}, (\omega(x, \cdot))_{x : x \cdot \ell \in [X_{\tau_i} \cdot \ell, X_{\tau_{i+1}} \cdot \ell)}\right\}, \qquad i \geqslant 1$$

is, under the measure $\mathbb{P}(\cdot|A_\ell^+)$, an i.i.d. sequence.

(b) Under the measure $\mathbb{P}$, the law of $\mathcal{V}_i$, conditioned on $\{A_\ell^+\}$, is identical to the law of

$$\left\{\tau_1, (X_n)_{n \leqslant \tau_1}, (\omega(x, \cdot))_{x : x \cdot \ell \in [0, X_{\tau_1} \cdot \ell)}\right\}$$

conditioned on the event $\{D = \infty\}$.

In words, the path of the RWRE between regeneration times, as well as the environment determined by hyperplanes perpendicular to ℓ visited between regeneration times, form an i.i.d. sequence under the event A_ℓ^+.

Lemma 5.1 may look at first surprising, since the τ_i's, being forward looking, are not stopping times. However, it turns out that all the information they convey is simply the fact that the starting time is a regeneration time. The formal proof is obtained by making explicit the last statement, we refer to [SzZr99] or [Zt04] for details.

If $\mathbb{E}(\tau_1|D = \infty) < \infty$, then also $\mathbb{E}(|X_{\tau_1}||D = \infty) < \infty$, and lemma 5.1 together with an interpolation argument shows immediately that, with

$$v = \frac{\mathbb{E}(X_{\tau_1}|D = \infty)}{\mathbb{E}(\tau_1|D = \infty)} \neq 0, \tag{5.1}$$

it holds that

$$\mathbb{P}\left(\frac{X_n}{n} \to_{n\to\infty} v|A_\ell^+\right) = 1.$$

On the other hand, if $\mathbb{E}(\tau_1|D = \infty) = \infty$, a renewal argument whose details can be found in [Zr98] shows that necessarily, $X_n/n \to 0$ on the event A_ℓ^+. Combined together, these facts prove part (a) of theorem 4.4.

5.1.3. The 0 − 1 law. As mentioned in theorem 4.3, when $d = 2$ and P is i.i.d. and elliptic, it holds that $\mathbb{P}(A_\ell^+) \in \{0, 1\}$. The proof, due to [ZrM01], proceeds as follows. Consider the function $v(x) = P_\omega^x(\lim_{n\to\infty} X_n \cdot \ell = \infty)$. Then, a martingale argument shows that $v(X_n)$ converges to 1 on A_ℓ^+. Now, assume that $\mathbb{P}(A_\ell^+)\mathbb{P}(A_\ell^-) > 0$. Then, a RWRE started at the origin has a positive probability to end up on the event A_ℓ^+, while a RWRE started any point $(L, y) \in \mathbb{Z}^2$ has a positive probability to end up on the event A_ℓ^-. By choosing properly the point y, one can ensure that with positive probability that does not depend on L, the two paths cross (here is where $d = 2$ enters: in higher dimension, this needs not be true). But at the point x they cross, it is impossible that $v(x)$ is close to 1, contradicting the convergence of $v(X_n)$ to 1 on A_ℓ^+.

We digress next and explain the construction of counter examples to the 0-1 law, for non i.i.d. environments. We begin by considering $d = 2$, following [ZrM01]. Consider the lattice $\mathbb{Z}^2$ and its sub-lattice $2\mathbb{Z}^2$. Connect each vertex in $2\mathbb{Z}^2$ to either its northern or eastern neighbour in that lattice, in an i.i.d., equally likely fashion. Extend this to a graph on $\mathbb{Z}^2$ in an obvious manner (thus, if $(0, 0)$ is connected to $(0, 2)$ in the sublattice $2\mathbb{Z}^2$, then $(0, 0)$ is connected to $(0, 1)$ and $(0, 1)$ is connected to $(0, 2)$ in the lattice $\mathbb{Z}^2$). The resulting graph is a tree $\mathcal{T}$ (marked by solid line in figure 3), and it is easy to check that almost surely, it has one connected component, and each vertex $x = (x_1, x_2)$ on the tree is connected to only finitely many vertices $y = (y_1, y_2)$ with both $y_1 \leqslant x_1$ and $y_2 \leqslant x_2$ (such vertices are called descendants of x). Let $l(x)$ denote the distance (on $\mathcal{T}$) between x and its farthest descendant. Let $a(x)$ denote the ancestor of x, which is the unique vertex connected to x that has x as descendant. Define $\omega(x, a(x)) = 1 - 1/l(x)^2$ and $\omega(x, e) = 1/3l(x)^2$ for $e \neq a(x)$. Finally, note that $\mathcal{T}$ defines naturally a dual tree $\mathcal{T}'$ with vertices in $\mathbb{Z}^2$, which 'points' in the opposite direction (this tree is constructed from vertices in $2\mathbb{Z}^d + (1, 1)$, that start by being connected either to their southern or western neighbour). The definition of descendant $y = (y_1, y_2)$ of $x = (x_1, x_2)$ is that $y_1 \geqslant x_1$ and $y_2 \geqslant x_2$. One defines $l(x)$ on $\mathcal{T}'$ in a similar fashion to $\mathcal{T}$. Finally, to make the construction stationary, one applies a random unit shift of the lattice $\mathbb{Z}^2$ along one of the coordinate axes, with equal probability among the four possible shifts.

Let $\ell = (1, 1)/\sqrt{2}$. Since $l(X_{n+1}) \geqslant l(X_n) + 1$ if X_{n+1} is an ancestor of X_n, the Borel–Cantelli lemma implies that a RWRE started on $\mathcal{T}$ has a positive probability to stay on $\mathcal{T}$

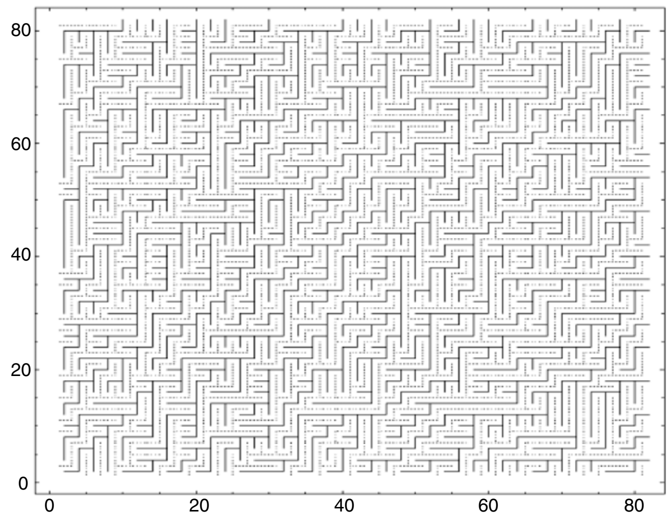

Figure 3. The trees $\mathcal{T}$ (solid line) and $\mathcal{T}'$ (dashed line).

forever and advance at each step towards an ancestor. In particular, on this event of positive probability, $X_n \cdot \ell \to \infty$ (and in fact, the motion is ballistic). On the other hand, if the RWRE starts on $\mathcal{T}'$ then it has a positive probability to satisfy $X_n \cdot \ell \to -\infty$. Since the RWRE also has, by ellipticity, a positive probability to move from $\mathcal{T}$ to $\mathcal{T}'$ and vice versa, we conclude that $\mathbb{P}(A_\ell^+)\mathbb{P}(A_\ell^-) > 0$. Thus, the 0-1 law cannot hold true for such an environment.

The construction described above yields a P which is neither mixing nor uniformly elliptic. For $d \geqslant 3$, both these points can be overcome, essentially by adding 'insulation' around the tree $\mathcal{T}$ and moving the tree $\mathcal{T}'$ away from $\mathcal{T}$ (the higher dimension is needed to allow for enough space for such separation). This leads to part (b) of theorem 4.3. The resulting environment, besides being uniformly elliptic, is polynomially mixing, i.e. exhibits polynomial decay of correlations (which is however not summable). The details of the construction can be found in [BrZZ06].

5.1.4. Ballistic behaviour: moment bounds and condition T'. A consequence of lemma 5.1 is that if $\mathbb{P}(A_\ell^+)$ holds true and also $\mathbb{E}(\tau_1|D = \infty) < \infty$, then by decomposing X_n and n into a sum of i.i.d. random variables (the regeneration increments $X_{\tau_{i+1}} - X_{\tau_i}$ and the time increments $\tau_{i+1} - \tau_i$) and a small remainder, it is easy to show that ballistic behaviour occurs. Further, as soon as also $\mathbb{E}(\tau_1^2 | D = \infty) < \infty$, then an annealed CLT holds true. Hence, the key to both the ballistic behaviour and the CLT lies in obtaining good moment bounds on the annealed law of τ_1 conditioned on the event $D = \infty$.

In [SzZr99], the authors proved under Kalikow's condition (4.3) that $\mathbb{E}(\tau_1|D = \infty) < \infty$. For doing so, they first observed that under (4.3),

$$\mathbb{E}(X_{T_U} \cdot \ell) \geqslant \epsilon_\ell \mathbb{E}(T_U), \tag{5.2}$$

allowing one to reduce the issue of tail estimates on τ_1 to the question of tail estimates on the displacement $X_{\tau_1} \cdot \ell$, conditioned on $D = \infty$. But, since every fresh point has a positive probability (under (4.3)) to be a regeneration point, and the backtrack distance for the RWRE has exponential moments, one concludes that $X_{\tau_1} \cdot \ell$ also possesses exponential moments. From this the conclusion follows.

Relation (5.2) cannot be directly extended to obtain higher moment controls on τ_1, and hence is not directly useful in proving the CLT. A breakthrough came with the work of Sznitman [Sz00], who showed how the exponential moment estimates on $|X_{\tau_2} - X_{\tau_1}|$ can be translated into estimates on τ_1, as follows.

Lemma 5.2. *Assume $d \geqslant 2$, P is uniformly elliptic and i.i.d., and Kalikow's condition (4.3) holds. Then, there exists some $\alpha > 1$ such that for all u large,*

$$\mathbb{P}(\tau_1 > u) \leqslant \mathrm{e}^{-(\log u)^{\alpha}}. \tag{5.3}$$

In particular, all moments of τ_1 are finite.

Recall that when $d = 1$, Kalikow's condition is equivalent to the condition $s > 1$. Contrasting lemma 5.2 with (3.3) shows that lemma 5.2 is not true for $d = 1$, which is another manifestation of the intuition that traps are weaker in high dimension than in low dimension.

Recall the variables $T_{\ell,b,L}$ introduced above definition 4.5. The proof of lemma 5.2 follows rather directly from Kalikow's condition, the ellipticity assumption and the estimate.

There exist $\beta < 1$ and $\xi > 1$ such that

$$\limsup_{L\to\infty} \frac{1}{L^{\xi}} \log P\left(P_{\omega}^{0}\left(X_{T_{\ell,1,L}} \cdot \ell \geqslant L\right) \leqslant \mathrm{e}^{-cL^{\beta}}\right) < 0, \tag{5.4}$$

The estimate in (5.4) is where most of the work is invested. It essentially gives an upper bound on the probability that a piece of the environment has strong blocking properties for the RWRE. Its proof is built on constructing 'channels' along which exit can occur (here, $d \geqslant 2$ is used), and arguing that if (5.4) were not true, then all these channels would have to block the walk, which is not possible since the channels are independent and Kalikow's condition gives a lower bound on the probability of a channel to be 'non-blocking'. We refer for details to [Sz00] and the expositions in [Sz04] and [Zt04].

Having discussed Kalikow's condition, we turn to Condition T'. Its usefulness is in the following result from [Sz03a].

Proposition 5.3. *Let $d \geqslant 2$, $\ell \in S^{d-1}$, and $\gamma \in (0, 1]$. The following are equivalent.*

(a) Condition T_{γ} holds.

(b) $\mathbb{P}(A_{\ell}^{+}) = 1$ and, with $X^{*} := \sup_{0\leqslant n\leqslant \tau_1} |X_n|$, there exists a $c > 0$ such that

$$\mathbb{E}(\exp(c(X^{*})^{\gamma})) < \infty.$$

The proof is detailed in [Sz03a]; see also the exposition in [Sz04]. From part (b) of proposition 5.3, tail estimates on τ_1 of the form (5.3) follow, by a route similar to that described above when discussing Kalikow's condition. We omit further details.

5.1.5. Extensions to mixing environments. When the law P on the environment is not i.i.d., lemma 5.1 fails to hold, and the usefulness of regeneration times is seriously limited. If the environment has a finite range of dependence (that is, there exists a K such that if $A \subset \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d$ satisfy $d(A, B) > K$, then the collections $(\omega(x, \cdot))_{x\in A}$ and $(\omega(x, \cdot))_{x\in B}$ are independent), one uses ellipticity to modify the definition of regeneration times and preserve independence. We refer to [Zt04] for details; see also [She02] for related results. On the other hand, if the environment only satisfies a strong mixing condition but not finite range dependence, this cannot be done. Still, one may define modified regeneration times with good enough mixing conditions that ensure that both the LLN and the CLT hold, under a uniform Kalikow-type condition. We refer to [CZ04] and [CZ05] for details, and to [RA03] for an alternative approach leading to the LLN for certain mixing environments, which relies on the environment viewed from the point of view of the particle.

5.2. Homogenization in balanced environments

The proof of theorem 4.13 follows the homogenization approach discussed in section 3.4 for $d = 1$. However, unlike the case of $d = 1$, here an explicit invariant measure viewed from the point of view of the particle cannot be found. Instead, one proves its existence and absolute continuity with respect to P from *a priori* estimates on invariant measures for periodized environments with large period, adapting to the discrete setup arguments of [PaV82]. Specifically, let L_ω denote the operator

$$L_\omega u(x) = \sum_{i=1}^{d} \omega(x, x+e_i)[u(x+e_i) + u(x-e_i) - 2u(x)].$$

For a bounded domain $E \subset \mathbb{Z}^d$, set

$$\|u\|_{E,d} = \left(\frac{1}{|E|}\sum_{x\in E}|u(x)|^d\right)^{1/d}.$$

One then has the following lemma.

Lemma 5.4. *There exists a constant $C = C(\epsilon, d)$ such that*

(a) (Maximum principle). For any $E \subset \mathbb{Z}^d$ bounded, any functions u and g such that

$$L_\omega u(x) \geqslant -g(x), \qquad x \in E$$

satisfy

$$\max_{x\in E} u(x) \leqslant C\, diam(E)|E|^{1/d}\|g\|_{E,d} + \max_{x\in\partial E} u^+(x).$$

(b) (Harnack inequality) Any function $u \geqslant 0$ such that

$$L_\omega u(x) = 0, \qquad x \in D_R(x_0), \tag{5.5}$$

satisfies

$$\frac{1}{C}u(x_0) \leqslant u(x) \leqslant Cu(x_0), \qquad x \in D_{R/2}(x_0).$$

The estimates in lemma 5.4 are an adaptation to the discrete setup of the Alexandroff–Bakelman–Pucci estimates from PDE theory, and are developed in [L85] and [KuT90], see also [Zt04]. They are useful in showing that the sequence of invariant measures of the environment viewed from the point of view of the particle satisfies a (uniform in the period) $L^{1+1/(d-1)}$ estimate, from which the existence of an invariant measure in the original random environment follows.

5.3. Cut times

We sketch the proof of theorem 4.14. Set $S = \sum_{i=1}^{d_1}(q(e_i) + q(-e_i))$, let $\{R_n\}_{n\in\mathbb{Z}}$ denote a (biased) simple random walk in $\mathbb{Z}^{d_1}$ with transition probabilities q/S, and fix a sequence of independent Bernoulli random variable with $P(I_0 = 1) = S$, letting $U_n = \sum_{i=0}^{n-1} I_i$. Denote by X_n^1 the first d_1 components of X_n and by X_n^2 the remaining components. Then, for every realization ω, the RWRE X_n can be constructed as the Markov chain with $X_n^1 = R_{U_n}$ and transition probabilities

$$\overline{P}^0_\omega(X^2_{n+1} = z|X_n) = \begin{cases} 1, & X_n^2 = z,\ I_n = 1 \\ \omega(X_n, (X_n^1, z))/(1-S), & I_n = 0. \end{cases}$$

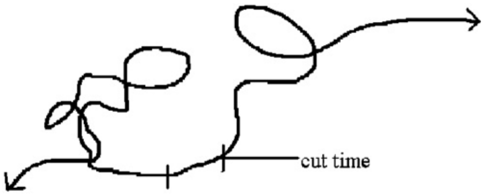

Figure 4. Cut times for the random walk R_n.

Introduce now, for the walk R_n, cut times c_i as those times where the past and future of the path R_n do not intersect; see figure 4. More precisely, with $\mathcal{P}_I = \{X_n\}_{n\in I}$,

$$c_1 = \min\{t \geqslant 0 : \mathcal{P}_{(-\infty,t)} \cap \mathcal{P}_{[t,\infty)} = \emptyset\}, \qquad c_{i+1} = \min\{t > c_i : \mathcal{P}_{(-\infty,t)} \cap \mathcal{P}_{[t,\infty)} = \emptyset\}.$$

The cut-times sequence depends on the ordinary random walk R_n only. In particular, because that walk evolves in $\mathbb{Z}^{d_1}$ with $d_1 \geqslant 5$, it follows from a Green function computation, as in [ET60], that there are infinitely many cut points, and moreover that they have a positive density. (We note in passing that due to the special role played by the origin, the differences $(c_{i+1} - c_i)$ are not stationary. However, they can be rendered stationary by making an appropriate change of measure, without modifying the asymptotic properties of the sequence.) The main observation is that the increments $X^2_{c_{i+1}} - X^2_{c_i}$ depend on disjoint parts of the environment. Therefore, conditioned on $\{R_n, I_n\}$, they are independent (with respect to the annealed measure $\mathbb{P}$). From here, the statement of theorem 4.14 is not too far.

5.4. From annealed to quenched CLT

Let $B^n_\cdot = (X_{[\cdot n]} - [\cdot n]v)/\sqrt{n}$ and let $\beta^n_\cdot$ denote the polygonal interpolation of $(k/n) \to B^n_{k/n}$. Let $C([0,T],\mathbb{R}^d)$ be endowed with the distance $d_T(f,g) = \sup_{s\leqslant T}|f(s) - g(s)| \wedge 1$. The following intuitively clear theorem, which is proved in [BoS02], is very useful in passing from annealed to quenched CLT's, especially in high dimension.

Theorem 5.5. *Suppose $B^n_\cdot$ satisfies the annealed invariance principle. Assume that for any $T > 0$, any bounded Lipschitz function F on $C([0,T],\mathbb{R}^d)$ (equipped with the distance d_T) and all $b \in (1,2]$,*

$$\sum_m \mathrm{Var}\big(E_\omega\big(F\big(\beta^{[b^m]}_\cdot\big)\big)\big) < \infty.$$

Then $B^n_\cdot$ satisfies the quenched invariance principle, i.e. for P-a.e. ω, $B^n_\cdot$ converges in distribution under P_ω to a deterministic scalar multiple of Brownian motion.

Theorem 5.5 was used in proving the quenched statements in theorem 4.14.

6. Multi-dimensional RWRE—the perturbative regime

We discuss in this section the perturbative analysis of the RWRE. By P being a small perturbation from a kernel q we mean that $q(\pm e_i) \geqslant 0$, $\sum_i [q(e_i) + q(-e_i)] = 1$, and for some ϵ small, $|\omega(x, x+e) - q(e)| < \epsilon$ for $e \in \{\pm e_i\}$. When $q(e) = 1/2d$ for $e = \pm e_i$, we say that P is a small perturbation from simple random walk.

We already observed, see remark 4.15, that in the perturbative regime for simple random walk, the RWRE can exhibit behaviour which is very different from the behaviour of simple random walk.

6.1. Ballistic walks

Sznitman's criterion for condition T' to hold (see theorem 4.9) together with an renormalization analysis allow one to give sufficient conditions for ballistic behaviour when ϵ is small. Set $\rho_0(3) = 5/2$ and $\rho_0(d) = 3$ for $d \geqslant 4$.

Theorem 6.1. *Let $d \geqslant 3$ and $\rho < \rho_0(d)$. Then there exists an $\epsilon_0 = \epsilon_0(d, \rho) > 0$ such that if P is i.i.d. and an ϵ perturbation from simple random walk, and $Ed_0 \cdot e_1 > \epsilon^\rho$, then the T' condition relative to e_1 holds.*

Theorem 6.1 appears in [Sz03a]. Contrasting its conclusion with the examples in remark 4.15 shows that some condition on the strength of the averaged drift Ed_0 as a function of ϵ is necessary. Also, $\rho_0(d) > 2$ is used in constructing the examples mentioned below theorem 4.9, which show that Kalikow's condition is strictly included in condition T'. We note that the case $d = 2$ is still open.

In another direction, if one writes $\omega(x, x + e) = q(e) + \epsilon\xi(x, x + e)$ with ξ i.i.d., and either $\sum eq(e) \neq 0$ or $\sum eq(e) = 0$ but $\sum eE\xi(0, e) \neq 0$, then for ϵ small enough, Kalikow's condition (4.3) holds. Expansions in ϵ of the speed of the RWRE are provided in [Sa03].

6.2. Balanced walks

Recall the balanced walks introduced in section 4.4 (cf theorem 4.13). The existence of an invariance measure viewed from the point of view of the particle, and the control achieved on this measure by approximations with periodized environments, allows one to get an expansion of the diffusivity matrix in terms of the strength of the perturbation from simple random walk. We refer the reader to [L89] for details.

6.3. Isotropic RWRE

The existence of sub-diffusive behaviour for the RWRE model in $d = 1$ immediately raises the question as to whether such sub-diffusive behaviour is present in higher dimension. As pointed out in section 4.2, see theorem 4.6, this is not the case when the environment satisfies condition T'. Since it may be expected that condition T' characterizes ballistic behaviour for $d > 1$, it is reasonable to expect (but not proved!) that for P i.i.d. and uniformly elliptic, and $d > 1$, no sub-diffusive behaviour is possible when the walk is transient in the direction ℓ (and further, in the ballistic regime, when recentring around the limiting velocity v, one expects fluctuations in the diffusive scale).

Outside the ballistic regime, rigorous results are few. Early attempts to address the question of the existence of a diffusive regime appeared in [DrL83, F84], using a formal renormalization group analysis in the small perturbation regime, with the conclusion that no sub-diffusive behaviour exists at $d \geqslant 3$ in the perturbative regime, and that at most logarithmic corrections to diffusive behaviour exist at $d = 2$. While this conclusion certainly conforms to what one would expect, soon after it was pointed out that counter-examples can be constructed (albeit not with i.i.d., or even finite range dependent, environments); see [BrD88, Br91, BGLd87]. Further, some of the examples discussed in this review, and in particular those of section 4.4, see remark 4.15, do not seem to be consistent with the formal renormalization analysis.

An attempt to put the analysis on a rigorous foundation was made in [BriKu91]. Among other things, they introduced the following isotropy condition.

Definition 6.2. *The law P on the environment is isotropic if, for any rotation matrix $\mathcal{O}$ acting on $\mathbb{R}^d$ that fixes $\mathbb{Z}^d$, the laws of $(\omega(0, \mathcal{O}e))_{e:|e|=1}$ and $(\omega(0, e))_{e:|e|=1}$ coincide.*

In particular, if P is isotropic then $Ed_0 = 0$. The main result of [BriKu91] is the following.

Theorem 6.3 (Bricmont–Kupiainen)**.** *Assume $d \geqslant 3$. There exists an $\epsilon_0 = \epsilon_0(d)$ such that if P is i.i.d. and isotropic, and an ϵ perturbation of simple random walk with $\epsilon < \epsilon_0$, then for some deterministic $\sigma^2 > 0$ and for P almost every ω, the sequence $X_n/\sigma\sqrt{n}$ converges in distribution, under P_ω^0, to a standard Gaussian random variable.*

The approach of [BriKu91] is to introduce a (diffusive) rescaling in time and space, and propagate an estimate on both the large scale behaviour of the RWRE, as well as about the existence of local traps that have the potential to destroy, at the next level, the diffusivity properties. The restriction to $d \geqslant 3$ is useful because the underlying simple random walk for $d \geqslant 3$ is transient, and hence Green function computations can be performed.

Unfortunately, the argument in [BriKu91] is hard to follow, and several attempts have recently been made to provide an alternative rescaling argument that is more transparent. The first approach [SzZ06], which is closest to theorem 6.3, has been undertaken in the context of diffusions in random environments, and consequently we postpone the discussion of it to section 7.3, even if this reverses the historical order in which results were obtained. In the remainder of this section, we describe another approach that yields a result concerning the exit measure of (isotropic) RWRE from large balls.

Let $V_L = \{x \in \mathbb{Z}^d : |x| \leqslant L\}$ be the ball of radius L in $\mathbb{Z}^d$ (where we recall that $|\cdot|$ is the euclidean norm), and let $\partial V_L = \{y \in \mathbb{Z}^d : d(y, V_L) = 1\}$ denote the boundary of V_L. Let $\tau_L = \min\{n : X_n \notin V_L\}$ denote the exit time of the RWRE from V_L, and for $x \in V_L$, $z \in \partial V_L$, let $\Pi_L(x, z) = P_\omega^x\big(X_{\tau_L} = z\big)$ denote the exit measure of the RWRE from V_L, and let $\pi_L(x, z)$ denote the corresponding quantity for simple random walk. Finally, let $\Pi_{L,l}^s(x, z) = \Pi_L \star \pi_{\eta l}$, where $\star$ denotes convolution and η is a random variable with smooth density supported on $(1, 2)$. $\Pi_{L,l}^s$ is a smoothed version of Π_L, where the smoothing is at scale l.

One expects that for an isotropic environment that is a small perturbation of simple random walk, the exit measure Π_L approaches that of simple random walk, except for small nonvanishing correction that are due to localized perturbations near the boundary, and that as soon as some additional smoothing is applied, convergence occurs. Under the assumptions of theorem 6.3, this is indeed the case. In what follows, for probability measures μ, ν we write $\|\mu - \nu\|$ for the variational distance between μ and ν.

Theorem 6.4. *Assume $d \geqslant 3$. There exists a $\delta_0 = \delta_0(d) > 0$ with the following property: for each $\delta < \delta_0$ there exists an $\epsilon_0 = \epsilon_0(d, \delta)$ such that if $\epsilon < \epsilon_0$ and P is an i.i.d. and isotropic law which is an ϵ perturbation of simple random walk, then*

$$\limsup_{L\to\infty} \|\Pi_L(0, \cdot) - \pi_L(0, \cdot)\| \leqslant \delta. \tag{6.1}$$

Further,

$$\limsup_{L\to\infty} \left\|\Pi_{L,l}^s(0, \cdot) - \pi_L \star \pi_{\eta l}(0, \cdot)\right\| \leqslant c_l \to_{l\to\infty} 0. \tag{6.2}$$

Theorem 6.4 is proved in [BoZ06]. We sketch the proof. Let $L_{n+1} = L_n(\log L_n)^3$. Write $\Delta_n = \Pi_{L_n} - \pi_{L_n}$. Considering the exit measures of the RWRE and simple random walk at scale L_{n+1} as those of coarse-grained walks with steps of size L_n (with an appropriate correction near the boundary), the perturbation expansion gives

$$\Pi_{L_{n+1}}(0, z) - \pi_{L_{n+1}}(0, z) = \sum_{k=1}^{\infty} [g_{n+1}\Delta_n]^k(0, y)\pi_{L_{n+1}}(y, z),$$

where g_{n+1} is the Green function of the simple random walk, coarse grained at scale L_n, and killed when exiting L_{n+1}.

Consider first the linear term $k = 1$, and write

$$\Pi_{L_{n+1}}(0, z) - \pi_{L_{n+1}}(0, z) = \sum_{w,y} g_{n+1}(0, w)\Delta_n(w, y)\big[\pi_{L_{n+1}}(y, z) - \pi_{L_{n+1}}(w, z)\big],$$

where we used that $\sum_y \Delta_n(w, y) = 0$. Restrict attention to y 'in the bulk', that is y such that $d\big(y, \partial V_{L_{n+1}}\big) > \delta L_{n+1}$. Then, by standard estimates for simple random walk, $\big\|\pi_{L_{n+1}}(y, z) - \pi_{L_{n+1}}(w, z)\big\| \leqslant (L_n/L_{n+1})$. On the other hand, since $d \geqslant 3$, $\sum_w g_{n+1}(0, w) = O((L_{n+1}/L_n)^2)$, which seems not good enough. However, one can use that the contribution of different w's that are not too close together in the sum $\sum_w g_{n+1}(0, w)\Delta_n(w, y)$ are independent, and of zero mean due to the isotropy condition. Using that shows that the linear term contributes a fixed but small contribution to the error in (6.1). Controlling the nonlinear term involves propagating an estimate of the form (6.2) from scale to scale, using the smoothing step in the perturbation expansion. This requires one to divide regions into 'good' (where this smoothing can be applied) and 'bad', aka traps (where smoothing cannot be propagated, but these regions are rare enough and hence, with high probability, not hit by the random walk). In fact, bad regions are classified according to four levels of badness, and some extra care needs to be exercised near the boundary when dealing with (6.1). We omit further details.

7. Diffusions in random environments

The model of RWRE possesses a natural analogue in the setup of diffusion processes.

7.1. One-dimensional generators

For dimension $d = 1$, the study of analogues of the RWRE model goes back to [Bx86] and [Sc85]. Formally, one looks at solutions to the stochastic differential equation

$$\mathrm{d}X_t = -\tfrac{1}{2}V'(X_t)\,\mathrm{d}t + \mathrm{d}\beta_t, \qquad X_0 = 0, \tag{7.1}$$

where β is a standard Brownian motion and V, the potential, is itself an (independent of β) Brownian motion with constant drift. Of course, (7.1) does not make sense as written, but one can express the solution to (7.1) for smooth V in a way that makes sense also when V is replaced by Brownian motion, by saying that conditioned on the environment V, X_t is a diffusion with generator

$$\frac{1}{2}\,\mathrm{e}^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{7.2}$$

The diffusion in (7.1) inherits many of the asymptotic properties of the RWRE model. Additional tools, borrowed from stochastic calculus, are often needed to obtain sharp statements. We refer to [Sh01] for details and additional references.

7.2. Multi-dimensional diffusions: finite range dependence

Like the RWRE in dimension $d = 1$, the model (7.1) leads to a reversible diffusion. A direct generalization of (7.1) via expression (7.2) for the generator, see for example [Ma94, Ma95], preserves the reversibility of the process, and thus for our purpose does not serve as a true analogue of the RWRE model. Instead, we consider diffusions satisfying the equation in $\mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t, \omega)\,\mathrm{d}t + \sigma(X_t, \omega)dW_t, \qquad X_0 = 0, \tag{7.3}$$

with generator

$$L = \frac{1}{2}\sum\nolimits_{i,j=1}^{d} a_{ij}(x,\omega)\partial_{ij}^2 + \sum\nolimits_{i=1}^{d} b_i(x,\omega)\partial_i, \tag{7.4}$$

where $a = \sigma\sigma^T$ is a d-by-d matrix and the coefficients a, b are assumed to satisfy the following:

Assumption 7.1

(a) The functions $a(\cdot,\omega)$ and $b(\cdot,\omega)$ are uniformly (in ω) bounded by K, with Lipschitz norm bounded by K, and a is uniformly elliptic, i.e. $a(x,\omega) - \kappa I$ is positive definite for some $\kappa > 0$ independent of x or ω.
(b) The random field $(a(x,\omega), b(x,\omega))_{x\in\mathbb{R}^d}$ is stationary with respect to shifts in $\mathbb{R}^d$.
(c) The collection of random variables $(a(x,\cdot), b(x,\cdot))_{x\in A}$ and $(a(y,\cdot), b(y,\cdot))_{y\in B}$ are independent when $d(A,B) > R$.

Part (a) of assumption 7.1 ensures that (7.3) possesses a unique strong solution. Part (c) of assumption 7.1 is a 'finite range dependence' condition. We continue to write P_ω for the quenched law of the trajectories of the diffusion.

Many of the results described in sections 4.1 and 4.2 have been proved also in the context of diffusions, when assumption 7.1 holds. We refer to [She03, Goe06, Scz05a, Scz05b] for details.

7.3. Isotropic diffusions in the perturbative regime

The analogue of the isotropy condition 7.1(b) in the diffusion context is the following.

Assumption 7.2 (Isotropy). *For any rotation matrix $\mathcal{O}$ preserving the union of coordinate axes of $\mathbb{R}^d$,*

$(a(\mathcal{O}x,\omega), b(\mathcal{O}x,\omega))_{x\in\mathbb{R}^d}$ has same law under P as $(\mathcal{O}a(x,\omega)\mathcal{O}^T, \mathcal{O}b(x,\omega))_{x\in\mathbb{R}^d}$.

The analogue of theorem 6.3 is the following.

Theorem 7.3. *Let assumptions 7.1 and 7.3 hold. Then, there exists a constant $\epsilon_0 = \epsilon_0(d,K,R)$ such that if $|a(x,\omega) - I| \leqslant \epsilon_0$ and $|b(x,\omega)| \leqslant \epsilon_0$, for all $x \in \mathbb{R}^d, \omega \in \Omega$, then for some deterministic $\sigma^2 > 0$, for a.e. ω, the sequence of random variables $X_t/\sigma\sqrt{t}$ converges in distribution to a standard Gaussian random variable.*

(A full quenched invariance principle also holds under the assumptions of theorem 7.3.)

We sketch briefly the multi-scale approach of [SzZ06] to the proof of theorem 7.3. It is based on controlling the (scaled) Hölder norm of the operator associated with the transition probability of the diffusion. More precisely, fix $\beta \in (0, 1/2]$, let $L_{n+1} = L_n^{1+\alpha}$ where α is a small (but fixed) constant. Define the Hölder norm

$$\|f\|_{n,\beta} = \sup_x |f(x)| + L_n^\beta \sup_{x\neq y}\left|\frac{f(x) - f(y)}{|x-y|}\right|,$$

and for an operator H, let $\|H\|_{n,\beta}$ denote the operator norm with respect to the Hölder norm. Let $R_n(x, \mathrm{d}y) = P_\omega^x\big(X_{L_n^2} \in \mathrm{d}y\big)$ and, with an appropriate sequence α_n, set $R_n^0(x,\mathrm{d}y) = P^x\big(W_{\alpha_n L_n^2} \in \mathrm{d}y\big)$. The heart of the proof is to compare a suitably truncated version of R_n, in Hölder norm, with R_n^0. This is achieved by a perturbation expansion in the same spirit as in section 6.3, where the control on Hölder norm replaces the smoothing step there. However, as explained in section 6.3, an important issue is the avoidance of strong traps, which are measured here in a way reminiscent of Condition T: namely, with $x \in \mathbb{Z}^d$, let $V_n(x) = x + [0, L_n]^d$, chop each face of the boundary of $V_n(x)$ into 5^{d-1} congruent and disjoint $d-1$-dimensional cubes, denoted by $C_i(x)$, and build d-dimensional cubes $C'_i(x)$ which are based on $C_i(x)$ and intersect $V_n(x)$ only on $C_i(x)$. With $V'_n(x) = x + [-L_n/4, 5L_n/5]^d$, declare the strength of the trapping effect at scale n at x with respect to I and starting positions $A_x \subset V_n(x)$ with a diameter of A_x less than L_{n-1}, as

$$J_{n,x,A_x,i} = \inf\left\{u > 0 : \inf_{y \in A_x} P_\omega^y\left(T_{C'_i} \leqslant L_n^2 \wedge T_{\partial V'_n(x)}\right) \geqslant c_1 L_n^{-\xi u}\right\},$$

with ξ and c_1 appropriately chosen constants, and for any set U, T_U is the hitting time of U by the diffusion. Then, the control on traps is achieved by the following inductive estimate: for any collection $\mathcal{A}$ of points $x \in L_n\mathbb{Z}^d$, and sets A_x as above, which are separated by distance at least $10dL_{n-1}$, and any i_x,

$$P(\text{for all } x \in \mathcal{A}, J_{n,x,A_x,i_x} \geqslant u_x) \leqslant L_n^{-\bar{M}_n \sum_{x \in \mathcal{A}}(u_x+1)},$$

with $\bar{M}_n$ an appropriate sequence that converges to a finite positive limit as $n \to \infty$. The propagation of the control of traps from scale to scale is done either by using the fact that strong traps are rare and hence rarely hit, or, for not so strong trap, similar to what was done in the ballistic case, i.e. constructing appropriate exit strategy for the diffusion to exit traps. We omit further details here, referring the reader to [SzZ06] instead.

8. Topics left out

We briefly mention in this section several topics that are related to this review but that we have not covered in detail.

8.1. Random conductance model

We have concentrated in this review on RWREs in i.i.d. environments, which give rise in the multi-dimensional case to non-reversible Markov processes. Although mentioned in several places, we did not discuss in detail the reversible case, where homogenization techniques using the environment viewed from the point of view of the particle are very efficient (note that the reversible case is a very particular case of an environment which is not i.i.d. but rather dependent with finite range dependence). The prototype for such reversible models is the 'random conductance model', where each edge (x, y) of $\mathbb{Z}^d$ is associated a (random, i.i.d.) conductance $\mathcal{C}_{x,y}$, and the transition probability between x and y is $\mathcal{C}_{x,y}\big/\left(\sum_{z:|z-x|=1} \mathcal{C}_{x,z}\right)$. Annealed CLTs for the random conductance model are provided in [Kun83, DFGW89]. See also [AKS82] for a related model with symmetric transitions. The quenched CLT is obtained in [Bov03] and [SdSz04].

One of the motivations to consider the random conductance model is the analysis of random walk on supercritical percolation clusters. The annealed CLT is covered by [DFGW89]. Several recent papers discuss the quenched case, first in dimension $d \geqslant 4$ [SdSz04], and then in all dimensions $d \geqslant 2$; see [BeBi06, MaPi05]. In another direction, when one discusses *biased* walks on a percolation cluster, new phenomena occur, for example the lack of monotonicity of the speed of the walk in the strength of the bias, which is again a manifestation of the trapping phenomenon. We refer to [BeGaP03] and [Sz03b] for details.

8.2. Brownian motion in a field of random obstacles

Another closely related (reversible) model is the model of Brownian motion in a field of obstacles in $\mathbb{R}^d$. Here, one defines a potential $V(x, \omega) = \sum_i W(x - x_i)$ where the collection $\{x_i\}$ is a (random) configuration of points in $\mathbb{R}^d$ (usually, taken according to a Poisson law) and W is a a fixed nonnegative shape function. Of interest are the properties of Brownian motion $(X_t)_{t\in[0,T]}$, perturbed by the change of measure

$$\Lambda_T = \frac{1}{Z_T(\omega)} \exp\left(-\int_0^T V(X_s, \omega)\, \mathrm{d}s\right).$$

It is common to distinguish between 'soft traps', with W bounded and typically of compact support, and 'hard trap', where $W = \infty \mathbf{1}_C$ where C is a given compact set. One is interested in understanding various path properties, as T gets large, or in understanding the quenched partition function $Z_T(\omega)$ and its annealed counterpart EZ_T. Due to reversibility, the problem is closely related to the study of the bottom λ_ω of the spectrum of $-\Delta/2+V$, and the difficulty is in understanding the structure of those traps that influence λ_ω. A good overview of the model and the techniques developed to analyse it, including the 'method of enlargement of obstacles', can be found in [Sz98].

8.3. Time-dependent RWRE

An interesting variant of the RWRE model has been proposed in [BdMP97]. In this model, the random environment is dynamic, i.e. changes with time, and so we write $\omega(x, x+e, n)$ where we wrote before $\omega(x, x+e)$. In the simplest version, the collection of random vectors $(\omega(x, x+\cdot, n))_{x\in\mathbb{Z}^d, n\in\mathbb{N}}$ is i.i.d. Annealed, the RWRE is then a simple random walk in an averaged environment, but the true interest lies in obtaining quenched statements. Those were obtained in [BdMP97, BdMP04] by a perturbative approach. An alternative, simpler proof is given by [Sta04]. Another approach to the quenched CLT, that covers other cases of random walk 'with a forbidden direction', is developed in [RASe05], based on a general pointwise CLT for additive functionals of Markov chains due to [DerLi03].

An interpolation between the RWRE model and the i.i.d. dynamical environment model is when the collection $(\omega(x, x+\cdot, n))_{x\in\mathbb{Z}^d, n\in\mathbb{N}}$ is i.i.d. in x but Markovian in n. This case has been analysed by perturbative methods in [BdMP00], and by regeneration techniques in [BaZt06]. In both cases, an annealed CLT holds in any dimension, but the quenched CLT was obtained only in high dimension. It is still open to determine whether in the Markovian setup, there are examples where the quenched CLT fails.

8.4. RWRE on trees and other graphs

We have already mentioned the interest in considering random walks on random subgraphs of $\mathbb{Z}^d$, and in particular percolation clusters. Of course, one may consider instead random walk (or biased random walk) on other random graphs. A particularly important class of models treats random walks on random trees, and in particular Galton–Watson trees. We refer to [LyP06] for an excellent overview of the properties and ergodic theory of such random walks, and to [PZe06] for recent results concerning the CLT. See also [HuSh06] for slowdown estimates for the analogue of the RWRE on the binary tree. We emphasize that these models are all reversible.

8.5. Non-nearest-neighbour RWRE

Many of the techniques described in this survey have a natural generalization to non-nearest-neighbour walks. In particular, the results in [V04, RA04] are already stated in terms of compactly supported transition probabilities, and the development of regeneration times can easily be extended, following the techniques in [CZ04, CZ05], to the non-nearest neighbour, finite range setup. However, to the best of my knowledge, no systematic study of RWRE for non-nearest neighbour RWREs in dimension $d \geqslant 2$ has appeared in the literature.

The situation is different in dimension $d = 1$, where the RWRE is no longer reversible. It was early realized, see [Ky84, Le84], that ergodic theorems involve the study of certain Lyapounov exponents associated with the product of random matrices. For some recent results, we refer to [BoGos00] and [Bre04b].

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References

[A99] Alili S 1999 Asymptotic behaviour for random walks in random environments *J. Appl. Prob.* **36** 334–49
[AKS82] Anshelevich V V, Khanin K M and Sinai Ya G 1982 Symmetric random walks in random environments *Commun. Math. Phys.* **85** 449–70
[BaZt06] Bandyopadhyay A and Zeitouni O 2006 Random walk in dynamic Markovian random environment *ALEA* **1** 205–24
[Be06] Berger N 2006 On the limiting velocity of high-dimensional random walk in random environment *Preprint* math.PR/0601656
[BeGaP03] Berger N, Gantert N and Peres Y 2003 The speed of biased random walk on percolation clusters *Probab. Theory Relat. Fields* **126** 221–42
[BeBi06] Berger N and Biskup M 2005 Quenched invariance principles for simple random walk on percolation clusters *Preprint* math.PR/0503576 (*Probab. Theory Relat. Fields* at press)
[BdMP97] Boldrighini C, Minlos R A and Pellegrinotti A 1997 Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment *Probab. Theory Relat. Fields* **109** 245–73
[BdMP00] Boldrighini C, Minlos R A and Pellegrinotti A 2000 Random walk in a fluctuating random environment with Markov evolution *On Dobrushin's Way: From Probability Theory to Statistical Physics (American Mathematical Society Transl. Ser. 2* vol 198*)* (Providence, RI: American Mathematical Society) pp 13–5
[BdMP04] Boldrighini C, Minlos R A and Pellegrinotti A 2004 Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive *Probab. Theory Relat. Fields* **129** 133–56
[BoGos00] Bolthausen E and Goldsheid I 2000 Recurrence and transience of random walks in random environments on a strip *Commun. Math. Phys.* **214** 429–47
[BoS02] Bolthausen E and Sznitman A S 2002 On the static and dynamic points of view for certain random walks in random environment *Methods Appl. Anal.* **9** 345–75
[BoZ06] Bolthausen E and Zeitouni O 2006 Multiscale analysis of exit distributions for random walks in random environments *Preprint* math.PR/0607192
[BoSZ03] Bolthausen E, Sznitman A S and Zeitouni O 2003 Cut points and diffusive random walks in random environments *Ann. Inst. Henri Poincare* **39** 527–55
[BGLd87] Bouchaud J P, Georges A and Le Doussal P 1987 Anomalous diffusion in random media: trapping, correlations and central limit theorems *J. Physique* **48** 1855–60
[Bov03] Boivin D and Depauw J 2003 Spectral homogenization of reversible random walks on $\mathbb{Z}^d$ in a random environment *Stoch. Proc. Appl.* **104** 29–56
[Br91] Bramson M 1991 Random walk in random environment: a counterexample without potential *J. Stat. Phys.* **62** 863–75

[BrD88] Bramson M and Durrett R 1988 Random walk in random environment: a counterexample? *Commun. Math. Phys.* **119** 199–211

[BrZZ06] Bramson M, Zeitouni O and Zerner M P W 2006 Shortest spanning trees and a counterexample for random walks in random environments *Ann. Probab.* **34** 821–56

[Bre04a] Brémont J 2004 Behavior of random walks on $\mathbb{Z}$ in Gibbsian medium *C. R. Math. Acad. Sci. Paris* **338** 895–8

[Bre04b] Brémont J 2004 Julien Random walks in random medium on $\mathbb{Z}$ and Lyapunov spectrum *Ann. Inst. Henri Poincare Probab. Stat.* **40** 309–36

[BriKu91] Bricmont J and Kupiainen A 1991 Random walks in asymmetric random environments *Commun. Math. Phys* **142** 345–420

[Bx86] Brox T 1986 A one-dimensional diffusion process in a Wiener medium *Ann. Probab.* **14** 1206–18

[Ch05] Cheliotis D 2005 Diffusions in random environments and the renewal theorem *Ann. Probab.* **33** 1760–81

[CGZ00] Comets F, Gantert N and Zeitouni O 2000 Quenched, annealed and functional large deviations for one dimensional random walk in random environment *Probab. Theory Relat. Fields* **118** 65–114

[CZ04] Comets F and Zeitouni O 2004 A law of large numbers for random walks in random mixing environments *Ann. Probab.* **32** 880–914

[CZ05] Comets F and Zeitouni O 2005 Gaussian fluctuations for random walks in random mixing environments *Isr. J. Math.* **148** 87–114

[DGPS05] Dembo A, Gantert N, Peres Y and Shi Z 2005 Valleys and the maximum local time for random walk in random environment *Preprint* math.PR/0508579

[DZ98] Dembo A and Zeitouni O 1998 *Large Deviations Techniques and Applications* 2nd edn (Berlin: Springer)

[DGuZ01] Dembo A, Guionnet A and Zeitouni O 2001 Aging properties of Sinai's random walk in random environment *Preprint* math.PR/0105215

[DGPZ02] Dembo A, Gantert N, Peres Y and Zeitouni O 2002 Large deviations for random walks on Galton–Watson trees: averaging and uncertainty *Probab. Theory Relat. Fields* **122** 241–88

[DGaZ04] Dembo A, Gantert N and Zeitouni O 2004 Large deviations for random walk in random environment with holding times *Ann. Probab.* **32** 996–1029

[DPZ96] Dembo A, Peres Y and Zeitouni O 1996 Tail estimates for one-dimensional random walk in random environment *Commun. Math. Phys.* **181** 667–84

[DerLi03] Derriennic Y and Lin M 2003 The central limit theorem for Markov chains started at a point *Probab. Theory Relat. Fields* **125** 73–6

[DrL83] Derrida B and Luck J M 1983 Diffusion on a random lattice: weak-disorder expansion in arbitrary dimension *Phys. Rev.* B **28** 7183–90

[DnV83] Donsker M D and Varadhan S R S 1983 Asymptotic evaluation of certain Markov process expectations for large time, IV *Commun. Pure Appl. Math.* **36** 183–212

[DoS84] Doyle P G and Snell J L 1984 *Random Walks and Electric Networks (Carus Mathematical Monographs* vol 22*)* (Washington, DC: Mathematical Association of America)

[ET60] Erdös P and Taylor S J 1960 Some intersection properties of random walks paths *Acta Math. Acad. Sci. Hungar.* **11** 231–48

[F84] Fisher D S 1984 Random walks in random environments *Phys. Rev.* A **30** 960–4

[Ga02] Gantert N 2002 Subexponential tail asymptotics for a random walk with randomly placed one-way nodes *Ann. Inst. Henri Poincare* **38** 1–16

[GaS02] Gantert N and Shi Z 2002 Many visits to a single site by a transient random walk in random environment *Stoch. Process. Appl.* **99** 159–76

[GZ98] Gantert N and Zeitouni O 1998 Quenched sub-exponential tail estimates for one-dimensional random walk in random environment *Commun. Math. Phys.* **194** 177–90

[Goe06] Goergen L 2006 Limit velocity and zero-one laws for diffusions in random environment *Preprint* math.PR/0512061 (*Ann. Appl. Probab.* at press)

[Gos06] Goldsheid I 2006 Simple transient random walks in one-dimensional random environment: the central limit theorem *Preprint* math.PR/0605775

[Go85] Golosov A O 1985 On limiting distributions for a random walk in a critical one dimensional random environment *Commun. Moscow Math. Soc.* **199** 199–200

[GdH94] Greven A and den Hollander F 1994 Large deviations for a random walk in random environment *Ann. Probab.* **22** 1381–428

[HuSh00] Hu Y and Shi Z 2000 The problem of the most visited site in random environment *Probab. Theory Relat. Fields* **116** 273–302

[HuSh06] Hu Y and Shi Z 2006 A subdiffusive behavior of recurrent random walk in random environment on a regular tree *Preprint* math.PR/0603363

[Hg96] Hughes B D 1996 *Random Walks and Random Environments* (Oxford: Oxford University Press)

[Ka81] Kalikow S A 1981 Generalized random walks in random environment *Ann. Probab.* **9** 753–68

[Ke86] Kesten H 1986 The limit distribution of Sinai's random walk in random environment *Physica* A **138** 299–309

[Ky84] Key E S 1984 Recurrence and transience criteria for random walk in a random environment *Ann. Probab.* **12** 529–60

[KRV06] Kosygina E, Rezakhanlou F and Varadhan S R S 2006 Stochastic homogenization of Hamilton–Jacobi–Bellman equations *Commun. Pure Appl. Math.* at press

[KKS75] Kesten H, Kozlov M V and Spitzer F 1975 A limit law for random walk in a random environment *Comput. Math.* **30** 145–68

[Ko85] Kozlov S M 1985 The method of averaging and walks in inhomogeneous environments *Russ. Math. Surv.* **40** 73–145

[Kun83] Kunnemann R 1983 The diffusion limit of reversible jump processes in $\mathbb{Z}^d$ with ergodic random bond conductivities *Commun. Math. Phys.* **90** 27–68

[KuT90] Kuo H-J and Trudinger N S 1990 Linear elliptic difference inequalities with random coefficients *Math. Comput.* **55** 37–53

[L85] Lawler G F 1982 Weak convergence of a random walk in a random environment *Commun. Math. Phys.* **87** 81–7

[L89] Lawler G F 1989 Low-density expansion for a two-state random walk in a random environment *J. Math. Phys.* **30** 145–57

[L91] Lawler G F 1991 *Intersections of Random Walks* (Basle: Birkhauser)

[LdMF99] Le Doussal P, Monthus C and Fisher D 1999 Random walkers in one-dimensional random environment: exact renormalization group analysis *Phys. Rev.* E **59** 4795–4840

[Le84] Ledrappier F 1984 *Quelques Propriétés Des Exposants Charactéristiques (Lecture Notes in Mathematics* vol 1097*)* (New York: Springer)

[Li85] Liggett T M 1985 An improved subadditive ergodic theorem *Ann. Probab.* **13** 1279–85

[LyP06] Lyons R and with Peres Y *Probability on trees and networks* Available at http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html

[DFGW89] De Masi A, Ferrari P A, Goldstein S and Wick W D 1989 An invariance principle for reversible Markov processes: applications to random motions in random environments *J. Stat. Phys.* **55** 787–855

[MRZ04] Mayer-Wolf E, Roitershtein A and Zeitouni O 2004 Limit theorems for one-dimensional transient random walks in Markov environments *Ann. Inst. Henri Poincare* **40** 635–59 (CE: Check it)

[Ma94] Mathieu P 1984 Zero white noise limit through Dirichlet forms, with applications to diffusions in a random medium *Probab. Theory Relat. Fields* **99** 549–80

[Ma95] Mathieu P 1985 Limit theorems for diffusions with a random potential *Stoch. Proc. Appl.* **60** 103–11

[MaPi05] Mathieu P and Piatnitski A 2005 Quenched invariance principles for random walks on percolation clusters *Preprint* math.PR/0505672

[Mo94] Molchanov S A 1994 Lectures on random media *(Lecture Notes in Mathematics* vol 1581*)* (New York: Springer)

[PaV82] Papanicolaou G C and Varadhan S R S 1982 Diffusions with random coefficients *Statistics and Probability: Essays in Honor of C. R. Rao* (Amsterdam: North-Holland) pp 547–52

[PZe06] Peres Y and Zeitouni O 2006 A central limit theorem for biased random walks on Galton Watson trees *Preprint* math.PR/0606625

[Pe07] Peterson J 2007 *PhD Thesis* University of Minnesota (forthcoming)

[PP99] Pisztora A and Povel T 1999 Large deviation principle for random walk in a quenched random environment in the low speed regime *Ann. Probab.* **27** 1389–413

[PPZ99] Pisztora A, Povel T and Zeitouni O 1999 Precise large deviation estimates for a one-dimensional random walk in a random environment *Probab. Theory Relat. Fields* **113** 191–219

[RA03] Rassoul-Agha F 2003 The point of view of the particle on the law of large numbers for random walks in a mixing random environment *Ann. Probab.* **31** 1441–63

[RA04] Rassoul-Agha F 2004 Large deviations for random walks in a mixing random environment and other (non-markov) random walks *Commun. Pure Appl. Math* **57** 1178–96

[RASe05] Rassoul-Agha F and Seppäläinen T 2005 An almost sure invariance principle for random walks in a space-time random environment *Probab. Theory Relat. Fields* **133** 299–314

[Rv05] Révész P 2005 *Random Walk in Random and Non-Random Environments* 2nd edn (Singapore: World Scientific)

[ReY99] Revuz D and Yor M 1999 *Continuous Martingales and Brownian Motion* 3rd edn (Berlin: Springer)

[Sa03] Sabot C 2004 Random walks in random environment at low disorder *Ann. Probab.* **32** 2996–3023

[Scz05a] Schmitz T 2005 Diffusions in random environment and ballistic behavior *Preprint* math.PR/0509554

[Scz05b] Schmitz T 2006 Examples of condition (T) for diffusions in random environment *Preprint* math.PR/0607293

[Sc85] Schumacher S 1985 Diffusions with random coefficients *Contemp. Math.* **41** 351–6

[She02] Shen L 2002 Asymptotic properties of certain anisotropic walks in random media *Ann. Appl. Probab.* **12** 477–510

[She03] Shen L 2003 On ballistic diffusions in random environment *Ann. Inst. Henri Poincare* **39** 839–76

[Sh01] Shi Z 2001 Sinai's walk via stochastic calculus *Milieux Aléatoires (Panoramas et Synthèses* vol 12*)* ed F Comets and E Pardoux pp 53–74 (Societe Mathematique de France)

[SdSz04] Sidoravicius V and Sznitman A S 2004 Quenched invariance principles for walks on clusters of percolation or among random conductances *Probab. Theory Related Fields* **129** 219–44

[Si82] Sinai Ya G 1982 The limiting behavior of a one-dimensional random walk in random environment *Theor. Prob. Appl.* **27** 256–68

[So75] Solomon F 1975 Random walks in random environments *Ann. Probab.* **3** 1–31

[Sp76] Spitzer F 1976 *Principles of Random Walk* 2nd edn (Berlin: Springer)

[StV79] Stroock D and Varadhan S R S 1979 *Multidimensional Diffusion Processes* (Berlin: Springer)

[Sta04] Stannat W 2004 A remark on the CLT for a random walk in a random environment *Probab. Theory Related Fields* **130** 377–87

[Sz98] Sznitman A S 1998 *Brownian Motion, Obstacles and Random Media* (Berlin: Springer)

[Sz99] Sznitman A S 1999 On a class of transient random walks in random environment *Ann. Probab.* **29** 724–65

[Sz00] Sznitman A S 2000 Slowdown estimates and central limit theorem for random walks in random environment *J. Emergency Med. Serv.* **2** 93–143

[Sz02] Sznitman A S 2002 An effective criterion for ballistic behavior of random walks in random environment *Probab. Theory Relat. Fields* **122** 509–44

[Sz03a] Sznitman A S 2003 On new examples of ballistic random walks in random environment *Ann. Probab.* **31** 285–322

[Sz03b] sznitman A S 2003 On the anisotropic walk on the supercritical percolation cluster *Commun. Math. Phys.* **240** 123–48

[Sz04] Sznitman A S 2004 Topics in random walks in random environment *School and Conference on Probability Theory (ICTP Lect. Notes* vol XVII*)* (Trieste: Abdus Salam Int. Cent. Theoret. Phys.) pp 203–266

[SzZr99] Sznitman A S and Zerner M 1999 A law of large numbers for random walks in random environment *Ann. Probab.* **27** 1851–69

[SzZ06] Sznitman A S and Zeitouni O 2006 An invariance principle for isotropic diffusions in random environments *Invent. Math.* **164** 455–567

[T72] Temkin D E 1972 One dimensional random walk in a two-component chain *Sov. Math. Dokl.* **13** 1172–6

[V66] Varadhan S R S 1966 Asymptotic probabilities and differential equations *Commun. Pure Appl. Math.* **19** 261–86

[V04] Varadhan S R S 2003 Large deviations for random walks in a random environment *Commun. Pure Appl. Math.* **56** 1222–45

[Zt02] Zeitouni O 2002 *Random walks in Random Environments: Proc. ICM 2002 (Documenta Mathematica* vol III*)* (Beijing: Higher Ed press) pp 117–27

[Zt04] Zeitouni O 2004 Random walks in random environment *XXXI Summer School in Probability (St. Flour, 2001) (Lecture Notes in Mathematics* vol 1837*)* (Berlin: Springer) pp 193–312

[Zr98] Zerner M P W 1998 Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment *Ann. Probab.* **26** 1446–76

[Zr02] Zerner M P W 2002 A non-ballistic law of large numbers for random walks in i.i.d. random environment *Electron. Commun. Probab.* **7** 191–7 (paper 19)

[ZrM01] Zerner M P W and Merkl F 2001 A zero-one law for planar random walks in random environment *Ann. Probab.* **29** 1716–32